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Choose Your Words: Refining What Counts as Mathematical Discourse in Students' Negotiation of Meaning for Rate of Change of Volume

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"CHOOSE YOUR WORDS": REFINING WHAT COUNTS AS MATHEMATICAL DISCOURSE IN STUDENTS' NEGOTIATION OF MEANING FOR RATE OF CHANGE OF VOLUME

by

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ABSTRACT

"CHOOSE YOUR WORDS": REFINING WHAT COUNTS AS

MATHEMATICAL DISCOURSE IN STUDENTS'

NEGOTIATION OF MEANING FOR RATE OF

CHANGE OF VOLUME

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Department of Mathematics Education

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The purpose of this study is to describe how university honors calculus students negotiate meaning and language for conceptually important ideas through mathematical discourse. Mathematical discourse has been recognized as an important topic by mathematics education researchers of various theoretical perspectives. This study is written from a perspective that merges symbolic interactionism (Blumer, 1969) with personal agency (Walter & Gerson, 2007) to assert that human choice reflects, but is not determined by, meanings that are primarily developed through social interaction. The process of negotiation of meaning is identified, described, and analyzed in the discourse of four students and their professor as they draw conclusions about the volume of water in a reservoir based on graphs of inflow and outflow. Video data, participant work, and transcript were analyzed using grounded theory and other qualitative techniques to develop three narrative accounts. The first narrative highlights the participants' use of personal pronouns and personal experience to negotiate meaning for the conventional mathematical terms "inflection" and "concavity." The second narrative describes how the participants' choices in discourse reflect an effort to represent both their mathematical and experiential understandings correctly as they negotiate language to describe critical "zero points." The third narrative describes the participants' process of mapping analogical language and meaning from the context of motion to the context of water in a reservoir. Analysis of these three narratives from the perspective of conventional and ordinary mathematical language suggests that the contextualized study of mathematics may provide students access to mathematical discourse if the relevant mappings between mathematical language and language from other appropriate contexts are made explicit. Analysis from the perspective of social speech (Piaget 1997/1896) suggests that specific uses of personal pronouns, personal experience, and *revoicing* (O'Connor & Michaels, 1996) may serve to invite students to become participants in mathematical discourse. An agency-based definition of mathematical discourse is suggested for application in research and practice.

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CHAPTER 1: INTRODUCTION

A class of university honors calculus students was asked to draw conclusions about the quantity of water in the Quabbin Reservoir in Massachusetts, given qualitative graphs of the inflow and outflow of water in the reservoir over the period of one year (Figure 1, Hughes-Hallett et al., 1994, p.325).

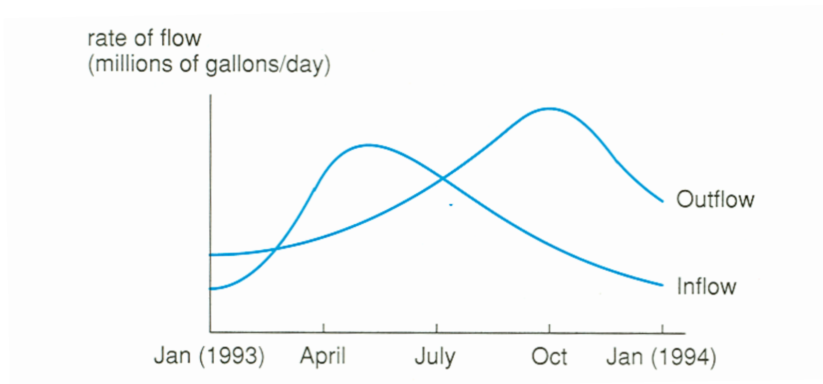


Figure 1. Graphs of inflow and outflow for the Quabbin Reservoir.

In the transcript below, one of the students, Daniel, explains the shape of his created graph of volume of water in the reservoir (Figure 2).

58	(0:13:50.1)	Daniel:	So, it has a negative slope. And then it starts going positive up to that point [July]. And so it levels off at zero. Cause the v-, the v- [1 sec] I don't know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. [2 sec] Which is correct on our velocity chart. And then it starts going negative again. And it starts, kind of, sloping out. And it has, its greatest slope is right here [October], so that's its inflection point

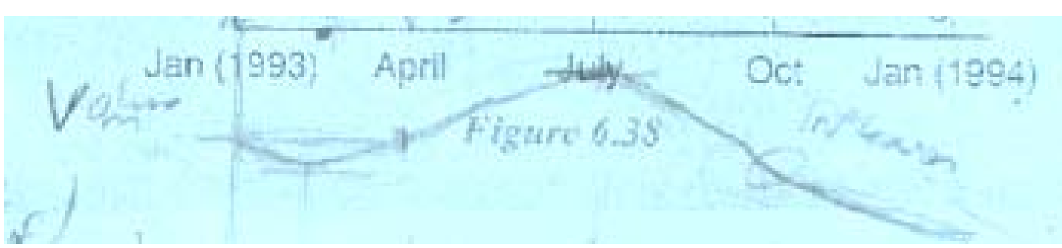


Figure 2. Daniel's created graph of volume.

As I investigated the mathematical discourse of these students and their instructor relevant to the Quabbin Reservoir Task, I had many questions about their choice of language. For example:

- How does a term like "velocity" end up in a conversation about water in a reservoir?
- What is the "it" of which Daniel speaks, how does he know when "it" is zero, and why is knowing that "it" is zero significant to Daniel?
- Is Daniel's switch between the third person "it" and the first person "I" reflective of notions of personal agency and social speech in mathematical discourse?
- What is the role of conventional terms such as "inflection point" in student-to-student discourse?

In this study, I view Daniel's statements above as a launching point for my investigation of these and other questions as I seek to answer the research question: "How do these university honors calculus students negotiate language and meaning for conceptually important ideas?"

The transcript above reflects a starting point for three narratives of the negotiation of meaning presented in this thesis. However, it would be difficult to characterize any piece of data as the official starting point or ending point of the *process* of negotiation of meaning. After Blumer (1969), I view meanings as social products, held by individuals who are constantly revising and refining meanings for things based on how other people act toward such things. As such, individual and collaborative processes of negotiation of meaning for things are ongoing and intertwined. For example, Daniel had previously heard the term "inflection point" in classroom discourse before using it in the transcript above. His current decision to use the term “inflection point,” and the way that he uses it, reflects his participation in, and interpretation of, that previous discourse. By introducing the term “inflection point” into the current discourse, Daniel reinitiates the social negotiation of meaning in which the participants question, explain, compare, agree, and disagree, reaching temporary closure as to how the conventional term “inflection point” will be used in their mathematical discourse. The first explanatory narrative in this thesis describes how Daniel’s initial use of “inflection point,” and the subsequent negotiation of meaning, is reflected in the language of another participant, Justin, as he provides an in-depth explanation of his perspective on the group’s goals and methods for the Quabbin Reservoir Task. In the second narrative, I describe how the participants negotiate language for what Daniel eventually calls “zero points”–points which are known to many as the conceptually important “critical points” in differential calculus. The third narrative investigates the participants’ negotiation of language and meaning for the analogical problem solving process that enables them to not only view, but speak of, the rate of change of volume as “the velocity of the water.”

Meanings and interpretations alone, however, cannot be said to determine human choices in discourse. Rather, humans exercise personal agency, acting upon their meanings and interpretations in ways that reflect the three explanatory and sometimes contradictory factors of (1) experience and imagination, (2) social roles and responsibilities, and (3) an individual’s concern for their own mathematical understandings (Walter & Gerson, 2007). The roles of these explanatory factors, as revealed through grounded theory and other qualitative research methods, are described in this study. I also characterize emergent views of conventional and ordinary mathematical language and Piaget's (1997/1896) notion of social and egocentric speech. This study contributes to literature on inquiry-based mathematics instruction by highlighting language factors involved in analogical problem solving. It contributes to research and theories of mathematical discourse by providing an in-depth analysis of the social process of negotiation of meaning and by suggesting an agency-based definition of mathematical discourse.

CHAPTER 2: THEORETICAL PERSPECTIVE

Viewing mathematical discourse as a conduit to greater understanding of mathematical meaning making and learning, I combine the perspective of symbolic interactionism (Blumer, 1969), with the notion of personal agency (Walter & Gerson, 2007) to frame my analysis of the mathematical discourse of four undergraduate students and their professor. After a brief explanation of the major premises of symbolic interactionism and personal agency, I introduce two continua that I use to describe and classify decisions made by these participants in mathematical discourse. These continua are characterized by the extremes of ordinary and conventional language (Walter & Johnson, 2007), and egocentric speech and social speech (Piaget, 1997/1896). Finally, although my view of agency suggests that decisions made by the participants are never completely determined by external factors, the perspective of symbolic interactionism suggests that specific explanatory factors do play a role in the participants' exercise of agency. I define three categories of explanatory factors for the purpose of this study.

Why Discourse?

Advocates of the *participationist* (Sfard, 2001) perspective expound upon the connection between thought and language with the "thinking as communication" metaphor. Rather than view mathematical knowledge as an entity to be acquired and kept independent of the context in which it is learned, participationists view learning as synonymous with becoming a participant in a given activity. To learn mathematics, therefore, is to become a participant in mathematical discourse (Sfard, 2001). *Aquisitionists*, on the other hand, view meaning, thought, and learning as separable products of mathematical activity. Nevertheless, most contemporary acquisition theories, including forms of constructivism, suggest that learning takes place as a result of social interaction (Cobb, 1994; Jones & Brader-Araje, 2002). Therefore, acquisitionists may regard discourse as a way of *getting at* mathematical meaning, thought, and learning, while the participationist viewpoint suggests that discourse may be the very embodiment of such things.

Whether these differing perspectives on learning are viewed as contradictory or complementary, both point to mathematical discourse as an important setting for studying mathematics learning (Sfard, 2001). Mathematics education researchers are coming to recognize what linguists have suggested for some time. As Goodwin (2000) put it:

> A primordial site for the analysis of human language, cognition, and action consists of a situation in which multiple participants are attempting to carry out courses of action in concert with each other through talk, while attending to both the larger activities that their current actions are embedded within, and relevant phenomena in their surround. (p. 1492)

Not only may discourse be viewed as an ideal setting for researchers to conduct their studies of mathematical language, cognition, and action, but recommendations for teaching practices have recently begun to suggest that mathematics may be best learned through greater student participation in collaborative problem solving and discourse (NCTM, 2000). Such recommendations, and the corresponding descriptions of ideal mathematical discourse, are helpful, but are also likely insufficient for the majority of mathematics teachers who are expected to implement such recommendations in their own practice. If discourse is to play such a central role in the mathematics classroom, teachers would be wise to have at least a working definition of what they think mathematical discourse should look like. As with many movements for educational change, the majority of mathematics teachers do not experience such practices first-hand as students, and therefore require rich descriptions that analyze rather than simplify. The National Council of Teachers of Mathematics (NCTM, 2007) has suggested that, “to effectively orchestrate mathematical discourse, teachers must do more listening, and students must do more reasoning” (p. 46). Likewise, I believe that researchers from all perspectives would benefit from an in-depth discussion of what conclusions can be drawn about individual and collective mathematical thought, simply by listening to students speak about mathematics. The intent of this study is to describe what kind of discourse mathematics students are not only capable of, but choose to, create and participate in as they engage in the mathematical activities of explanation and justification. I also suggest how this discourse may be reflective of, and contribute to, an underlying process of students learning to think mathematically as the participants comment on their discursive choices within the activity of justifying and refining their created solution for a mathematical problem.

It should be noted that my intent in describing mathematical discourse is not to provide teachers with an organized checklist or line-by-line script for an ideal form of mathematical discourse. I believe that asking learners of mathematics to participate in a scripted discourse for the purpose of building mathematical understanding may be a self-defeating enterprise. As they become participants in mathematical discourse, learners must have the opportunity and responsibility to act in ways that they believe will benefit and reflect their own mathematical understandings. As Goodwin (2000) suggests, such discourse should not (and I would suggest cannot) occur in a mathematical vacuum that is void of social context. Because it involves the exercise of human agency and varied human experiences and meanings, such discourse may initially seem disorganized and unproductive to the teacher or mathematician who has spent a large amount of time learning how to present mathematical arguments in logical and organized manners. What I intend to do is highlight the logic and progression of mathematical ideas that arise as a result of the participants' interactions and personal choices in mathematical discourse by focusing on the aspect of discourse that I characterize as "the negotiation of meaning."

Mathematical Discourse

Before progressing further into the theoretical perspectives of symbolic interactionism and personal agency, I offer *my* working definition of mathematical discourse, which will be refined and refocused through data analysis and the discussion of results in later chapters. Concise definitions of mathematical discourse are rare in the literature, and even the lengthier definitions often fail to explain under what conditions discourse should be considered mathematical (Moschkovich, 2003). The NCTM (2007) has stated that "the discourse of the learning community refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use as they engage in mathematical thinking and learning" (p.16). One with a robust vision of "mathematical thinking and learning" might be satisfied with this definition of discourse, but those who are still developing theory and practice might infer from such broad strokes that mathematical discourse refers to "anything happening in a mathematics classroom." Those with opposing visions of mathematical thinking and learning would likewise result in opposing visions of mathematical discourse. I believe that a more focused definition of mathematical discourse should be developed by mathematics educators, one that both communicates theory and directs practice.

Sfard (2001) describes two factors that one must confront in becoming a participant in mathematical discourse. The first factor is the set of *mediating tools,* such as language, representations, and ways of symbolizing, that are common to forms of discourse that are considered mathematical. Indeed, one of the most salient features for recognizing "mathematical" discourse may be the presence of specific mathematics symbols and terminology. Although the presence of mathematical terminology and representations in discourse may be correlated with mathematical activity and thought, I do not consider it a necessary or sufficient condition for defining mathematical discourse. In fact, the mere presence of conventional mathematics terminology is inconclusive evidence until further information is gathered regarding the function of such terminology in discourse. The use of mathematical terminology for mathematical enterprises such as proof, explanation, or generalization may constitute mathematical discourse. However, the use of unconventional terminology for mathematical enterprises may also constitute mathematical discourse. To determine whether discourse is mathematical, one must look beyond the form of language to also consider function, or how language is used (Halliday, 1978).

Sfard's (2001) second factor for becoming a participant in mathematical discourse is a set of *meta-discursive* rules that describe forms of communication that can be considered mathematical. When participationists speak of becoming a participant in mathematical discourse they refer not necessarily to memorizing a set of terms or definitions, but rather becoming a participant in a cultural practice. Gee's (1996), notion of “Discourses” (with a capital D) includes more than sentence structure and vocabulary (issues of form), extending to “ways of behaving, interacting, [and] valuing. . .” (p. viii) that may be considered mathematical within a culture. Following Gee’s notion of Discourses, a description of mathematical Discourse would necessarily include a description of mathematical behaviors, mathematical ways of interacting, and mathematical values.

Unfortunately, “mathematical” ways of behaving, interacting, and valuing are not well defined. For example, Richards (1991) identified four major domains of mathematical discourse, that of (1) research mathematicians and scientists, (2) mathematically literate adults in their daily lives, (3) mathematical print journals, and (4) mathematics classrooms. These four domains of discourse embrace different ways of behaving, valuing, and interacting due to their individual goals. However, Richards was nevertheless able to identify these four domains of discourse as mathematical. Therefore, one might assume the existence of an underlying definition of mathematical activity or a common set of mathematical values. For example, Moschkovich (2003) suggested that the values of mathematical discourse might include precision, explicitness, certainty, abstraction, and generalization.

For the purpose of this study, I define discourse as connected acts of speaking, gesturing, or symbolizing. By connected acts, I mean that individual acts are related to past acts or in anticipation of future acts. For discourse to be considered mathematical, (1) the content or topic must be mathematical objects, operations, or properties, and (2) the discourse must involve mathematical processes such as reasoning, explaining, conjecturing, and justifying. I impose two criteria for determining mathematical discourse because I recognize that one may speak *of* mathematical objects without engaging in mathematical processes (for example, consider the Initiation-Reply-Evaluation pattern described in the next chapter). On the other hand, a teacher may engage their students in reasoning (a process common to mathematical discourse) about a topic that is not mathematical (such as asking students to consider and explain why it is important to raise one's hand before speaking), and thus the resulting discourse would not be considered mathematical. When I speak of mathematical discourse, both criteria must be met. For example, teachers and students explaining and considering why the multiplication table has the structure that it does would be considered mathematical discourse.

Meaning and Symbolic Interactionism

As this study focuses on how the participants negotiate meaning through discourse, the meaning of *meaning* is also quite relevant. Terms used in discourse may be considered to have specific meaning for, or to be interpreted in differing ways by, the various participants in discourse. Although many constructs of meaning are virtually impossible to observe, it is possible to observe vocabulary usage, or how a learner chooses to use terminology in discourse (Dörfler, 2000). Therefore, from the viewpoint of an observer of discourse, it may be also much more productive to consider meaning a matter of function rather than form. The theory of symbolic interactionism (Blumer, 1969), which is described in further detail below, suggests that individual participants evaluate the way in which they believe terminology is or should be used in discourse and compare and contrast their own expectations with how terminology is used in the present discourse. Thus, participation in mathematical discourse involves participation in a negotiation of how terminology is and should be used. From my viewpoint as a researcher and observer, this negotiation of terminology usage in discourse is an approximation of the negotiation of mathematical meaning.

The three major premises of symbolic interactionism are: (1) “Human beings act toward things on the basis of the meanings that the things have for them;” (2) “the meaning of such things is derived from, or arises out of, the social interaction that one has with one’s fellows;” and (3) “these meanings are handled in, and modified through, an interpretative process used by the person in dealing with the thing he encounters” (Blumer, 1969, p. 2). Blumer differentiates the methodological position of symbolic interactionism from that of his contemporary psychologists and sociologists in two major areas. Those two areas are, first, perspectives on meaning and second, perspectives on human action.

First, the theory of symbolic interactionism is at odds with theories that suggest that meaning is either purely intrinsic or purely psychical. The meaning of an entity does not “belong” to the entity itself, nor is meaning purely dependent upon an individual’s perception of that thing. Rather, meaning *develops* in the context of social interaction. “The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. Their actions operate to define the thing for the person” (Blumer, 1969, p. 4-5). Therefore, meanings are viewed as flexible social products that reflect an individual’s interpretation of social interaction.

I might add that, while Blumer focused on the meaning of things as growing out of how other persons act toward a thing, it is also possible for persons to interact with things in a way that develops meaning. A simple example may be the meaning of a chair. While one might observe other people sitting on a chair and commenting upon the comfort of that chair, this does not prohibit an individual who has observed such actions from sitting on the same chair and, based upon their own bodily sensation, declare that chair to be uncomfortable and choose to sit on the floor. Therefore, a person's meaning for a thing may also grow out of that person's interaction with such things, as well as how other individuals act toward those things.

Blumer (1969) further differentiates symbolic interactionism from other theories of his day with respect to the principal explainer of human action. He recognizes that, although his contemporaries may not disagree with his assertion that human action is based in meaning, they often dismiss meaning as a minor or irrelevant factor when explaining human behavior. The psychologists, states Blumer, prefer to focus on external factors such stimuli, attitudes, and conscious or unconscious motives, while the sociologists focus on social positions, roles, norms, and values. Without denying the existence of such factors, the perspective of symbolic interactionism maintains meaning as the principal explainer of human behavior.

It is important to note that I also speak of meaning as an *explainer* of human action, rather than a *determiner* of human action. Meaning as a social product does not contribute directly back to human action and society without first passing though the channels of interpretation and the exercise of personal agency. According to Blumer (1969), the interpretive process by which meaning contributes to action involves two parts. First, in a process of self-communication, the person must indicate the object toward which they are acting. Second, the person "selects, checks, suspends, regroups, and transforms meanings in the light of the situation in which he is placed and the direction of his action" (p. 5).

For example, a participant in mathematical discourse may notice that another participant uses the term “limit” in a very different way than that which they have encountered in their own experience. Reflecting upon their own experience with the term “limit,” and their interpretation of this new use of “limit,” the participant may modify their own meaning to accommodate (Piaget & Inhelder, 2000/1969) this new meaning, or usage for the term. On the other hand, the participant may assume that they have encountered a case of homophony (two or more unrelated meanings for the same word), and assimilate (Piaget, & Inhelder, 2000/1969) the new use as a new or special case. The participant may also pose questions about others’ uses of the same terminology, avoid using a term which seems to have ambiguous meaning, suggest alternative terminology, or any other of a number of actions, each of which will likely contribute to further modification of meaning for the term “limit.”

Personal Agency

This ultimate choice of how to act, although based in previous action and interpretation of action, is determined by the exercise of personal agency. Although not specifically mentioned by Blumer (1969), I believe that personal agency not only explains, but ultimately determines, human action. Or, in other words, although a person’s formative process of interpreting meanings may guide their action in a specific direction, a person ultimately maintains the right to determine his or her action. Bandura (1989) suggests that human agency is “emergent interactive,” meaning that action, personal factors, and the environment influence one another in a process of “triadic reciprocal causation” (p.1175). Ahearn (2001) gives a provisional definition for agency as “the socioculturally mediated capacity to act” (p. 112). After Walter and Gerson

(2007), I define personal agency in mathematical activity as the “requirement, responsibility and freedom to choose based on prior experiences and imagination, with concern not only for one’s own understandings of mathematics, but with mindful awareness of the impact one’s actions and choices may have on others” (p. 209; see also Levinas, 1979; Martin, Sugarman & Thompson, 2003).

Embedded in this definition of personal agency (Walter & Gerson, 2007) are references to explanatory factors for human action, namely, (1) one’s prior experiences and imagination, (2) one’s current meaning for the mathematics, and (3) the consideration of the impact of one’s decisions upon others. As these three explanatory factors may compel a human to act in various and contradictory ways, none can be considered the determiner of human action. The capacity and responsibility of each participant to ultimately choose their action is the single factor that can be considered the determiner of human action. In other words, choices in mathematical discourse are determined by individuals, although they may be explained by the meanings that those individuals have developed for their experiences, the mathematics, and their fellow participants.

Some choices have greater impact on the resulting mathematical discourse than others. One important choice is the form of participation. Although all are referred to as “participants” in this study, some participants choose to participate in discourse in different ways than others. For example, at differing times, some participants choose to listen, some choose to question, some choose to explain, some choose to check for understanding, some choose to tell jokes, some choose to laugh at those jokes, and some choose to create representations. It is important to note that the choice to not participate is also a choice that contributes to the resulting mathematical discourse. However, this study focuses more specifically upon the choices that may be considered active participation in mathematical discourse, or the use of language and representations in ways that contribute to the negotiation of meaning among the participants.

Two Continua for Characterizing Participant Choices in Discourse

In this study, the participants' choices of language and non-verbal representations in mathematical discourse are described in terms of two continua. The first continuum is characterized by the extremes of conventional and ordinary language (Brown, 2001; Pirie, 1998; Walter & Johnson, 2007). Conventional language of mathematics includes technical terms along with their definitions and usages that are unique to the study of mathematics. These terms contribute greatly to the study of mathematics as a social practice (Pirie, 1998). Those who have appropriated the conventional language of mathematics may use it to identify abstract mathematical concepts quickly and precisely. Conventional mathematics terminology is often found in mathematics textbooks and documents that delineate core mathematical standards for mathematics educators. Mathematical dictionaries and glossaries also give mathematical definitions for conventional mathematics terminology.

Although much effort has gone into the creation of conventional mathematics definitions and terminology, few words can be categorized as belonging strictly to the realm of mathematics (Halliday, 1978). (For example, consider the terms "limit" and "set.") While it is true that language does not have to be conventional in order to function as mathematical, the value that certain communities of mathematical discourse attribute to conventional language makes such terminology a relevant issue of study. Certain kinds of non-verbal representations, as well, can be considered conventional within the realm of mathematical discourse, while others might be considered ordinary representations fulfilling mathematical roles in discourse. I do not see either of these extremes as more valuable than the other; rather, I view them as filling different roles. Where conventional language may allow entrance into specific communities of discourse and has been designed for purposes of precision and efficiency, ordinary language may be more appropriate for the process of “linguistic invention towards producing structures and meaning” (Brown, 2001, p. 76; see also Johnson, 2005; Walter & Johnson, 2007).

The second continuum for characterizing mathematical language and representation contrasts Piaget’s (1997/1896) notions of egocentric and social speech. Egocentric speech is characterized as speech in which the speaker “does not attempt to place himself at the point of view of his hearer” (p. 9). Children have been observed to exhibit egocentric speech through repetition, monologue, and collective monologues. Social speech, on the other hand, is directed toward a hearer. The speaker attempts to determine whether he has been understood, or attempts to interact with others in some manner. The speaker also adapts information in order to influence designated individuals to do or believe certain things. Piaget viewed criticism, requests, questions and answers as examples of social speech in children. Developmentally, Piaget believed egocentric speech to be a precursor to social speech. Vygotsky (1986/1934) also characterized egocentric speech and social speech, but suggested that egocentric speech was the developmental result of participating in social speech.

Bartsch and Wellman (1995) studied the development of children’s conceptions of the mind, noting that children under the age of three years not only described their own action in terms of their own desires and beliefs, but were also able to explain the action of other individuals in terms of those individuals' supposed desires and beliefs. Therefore, even young children have exhibited the capability of reasoning and speaking about others' points of view. Wertheimer (1945) described how a twelve year old boy examined a game of badminton from his losing opponent's point of view, and redefined the purpose of the game so that both players could enjoy the outcome. Wertheimer also noted how a tendency to only describe relationships from one's own point of view was reflected in a working woman's egocentric description of power relationships in the office where she worked.

Wertheimer (1945) described how forms of egocentric speech may be reflective of egocentric ways of viewing the world. "Productive thinking," he suggested, may involve "reasonable reorganization, [or] reorientation, which enables the subject to view the given situation in a new and more penetrating perspective" (p. 124). As my interest in egocentric and social speech is based in my efforts to better define the negotiation of meaning in discourse, egocentric discourse, or the participants' failure to consider or adopt the point of view of the hearer, may not be productive toward negotiating collaborative meaning within a group of students. On the other hand, forms of egocentric speech may help individual students to refine their own opinions and meanings for the mathematics by putting those meanings into words, and consequently contribute to the social negotiation of meaning. Therefore, while I do not consider either form of speech more productive or valuable than another, I believe that social and egocentric forms of speech may serve different purposes in mathematical discourse.

I view these two continua (conventional vs. ordinary and egocentric vs. social) as possible instruments for describing choices of language and representation made by participants in mathematical discourse. One might say that a participant in mathematical discourse exercises personal agency in choosing to use conventional language or ordinary language in either social speech or egocentric speech. However, this is not an exhaustive list of the available options. It is important to note that I have chosen my two continua as preliminary tools for characterizing choices made in discourse. While these choices are ultimately determined by the participants' exercise of personal agency, these choices can often be explained in terms of the interplay between constantly changing meanings for the mathematics, the language, and the various participants involved in the discourse. Naturally, the definition of discourse as "connected acts" implies that previous choices in discourse contribute to each participant's personal negotiation of meaning, and therefore can also be viewed as explanatory of subsequent choices in discourse.

Explanatory Factors for Choices in Mathematical Discourse

As mentioned before, three explanatory factors suggested by Walter and Gerson (2007) for the exercise of personal agency in mathematical discourse are (1) experience and imagination, (2) social roles and responsibilities, and (3) an individual's concern for their own mathematical understandings. I view these explanatory factors as representative of three categories of meaning, and refer to Goodwin (2000) and Bandura (1986) to further develop these categories. Goodwin states that although participants in discourse may be participating in an activity with particular goals, they also exhibit an awareness of the "larger activities that their current actions are embedded within" and the "relevant phenomenon in their surround" (p. 1492). For our participants, the "larger activities" in mathematical discourse may be the participants' meaning for the social roles and responsibilities that have been, and currently are, being negotiated through social interaction. Goodwin's mention of "relevant phenomena" supports the category of experience and imagination, as the participants build upon their personal experience in a physical world to reason about mathematical concepts. The relevance of social and experiential factors to all forms of discourse suggests that even mathematical discourse does not, and should not, resemble the precise, definite, and context-free communication of ideas that Descartes originally praised mathematics for when he described it as "mathesis universalis" (Cottingham, 1993, p. 7).

Bandura (1986) identifies four "determinants" for language development and use that are also related to the three categories of explanatory factors suggested by Walter and Gerson (2007). Bandura's four factors are (1) cognitive skills of the linguistic type (grammatical rules, the ability to abstract rules from exemplars. etc.), (2) non-linguistic knowledge (knowledge of the topic of the conversation, the ability to judge word meaning based on context, etc.), (3) the complexity of linguistic input (which may be too high or too low for optimum development), and (4) interpersonal factors (such as an individual's wishes to engage in social interaction or influence the social environment and that individual's interpretation of the results of his or her attempts to use language to do so). These factors may come to play in various combinations to explain the nature of the choices made in mathematical discourse. For example, although a mathematician may have been exposed to primarily conventional language in his mathematical career (complexity of linguistic input), he would be considered wise and pragmatic if he adjusted his language for a general interest lecture on number theory (interpersonal factors). An audience member at this lecture may be particularly familiar with a context in which a number theory problem is posed (non-linguistic knowledge), and therefore engage in productive mathematical discourse with the mathematician which provides that mathematician with valuable insight for a challenging problem.

To align Bandura's (1986) four factors with three explaining factors gleaned from Walter and Gerson (2007), I view Bandura's description of "non-linguistic knowledge" as a possible contributor to both the first factor of experience and imagination, and the third factor of an individual's understanding of the mathematics. For example, the use of reasoning tools such as analogy, metaphor, and examples to explain mathematical concepts in discourse may be explained by these two factors' incorporation of "non-linguistic knowledge." Bandura's "interpersonal factors" fit well into the second explanatory factor, which situates mathematical discourse in a social sphere. This factor may explain the participants' choices of egocentric or social speech, and, in the case of social speech, may explain just how much ordinary language or conventional language a speaker may choose to use to best negotiate meaning with the hearer. Bandura's "complexity of linguistic input" also contributes to the first explanatory factor of experience, as language choices often reflect previous uses of language.

While Bandura's (1986) mention of "cognitive skills" does not map directly to one of the agentive explanatory factors, the importance of such a factor to mathematical discourse cannot be denied, as one's possible choices in mathematical discourse may be shaped by a participant's own language abilities, or even possibly by a participant's knowledge of the structure of the language within which the discourse is taking place. In this study, I do not attempt to characterize the participants' linguistic knowledge or skills in any general sense. However, I do view improvisation (Holland, Lachicotte, Skinner, & Cain, 1998) for linguistic purposes as an example of a cognitive skill of the linguistic type that is related to the “imagination” component of the first explanatory factor. Improvisation may also be considered a meta-component of the explanatory factor framework, as it reflects an individual’s choice based on differing, and often contradictory meanings for several explanatory factors. In this study, instances of analogical problem solving and linguistic invention serve as examples of how participants combine their experience with mathematical meaning to develop new ways to speak about mathematics. Although improvisation is one of many possible cognitive skills that Bandura may have intended, it is particularly relevant to this study because Holland et al. described the appropriation of the results of improvisations as tools of agency and change.

The realization of the influence of the preexisting language structures on current mathematical discourse leads into a concluding discussion of the relationship between humans and language. Hermeneutics (Brown, 2001) suggests that the way humans perceive their world is influenced by the way in which they describe their world, and vice versa. Historically, mankind’s perception of the world, and perhaps the world itself, has evolved as a result of this hermeneutic cycle. Holland et al. (1998) described a similar cycle, stating that, “social scientists, today as in the past, are studying what their field of study has helped to create” (p. 24). In the next chapter, I describe how research on discourse in mathematics education reflects this hermeneutic cycle. Here, however, I consider the implications for the development of mathematical language and thought.

Brown (2001) describes two divergent viewpoints that exist within hermeneutics. The first is that humans can operate on language to mold their own perceptions over time, and may eventually come to interpret the world in a “correct” or desirable manner. The second viewpoint suggests that the reality of mankind is such a product of language that the hermeneutic cycle is beyond human control. This latter viewpoint may suggest that the language that has been previously chosen to describe mathematics has sent us on a trajectory of mathematical understanding that will continue to evolve despite any efforts we may make to redirect our progress. While I do not attempt to suggest the existence of a "perfect mathematics," I do maintain that the principle of personal agency remains the sole determiner of human action, including human language, speech, and thought. Inden (1990) defined human agency as "the realized capacity of people to act upon their world and not only to know about or give personal or intersubjective significance to it," suggesting that human agents not only have the capacity to reiterate the world through agency, but also the capacity to remake the world (p. 23; in Holland et al., 1998, p. 40). The previous choices of the human race have shaped our current experience in a system of language and expression that cannot, and should not, be ignored in mathematical discourse. By the same token, it is us, the participants in current mathematical discourse, who will create and direct mathematical language and thought in our present and future.

CHAPTER 3: REVIEW OF RELATED LITERATURE

In this chapter, I first review the progression of literature that describes and defines mathematical discourse and examine how such literature reflects shifting theoretical and practical perspectives on mathematics, mathematical activity, and mathematics education. In a second section, I review literature related to the content area of calculus, specifically the role of personal experience in graph interpretation and the development of meaning for the concept of derivative. The emphasis on the role of personal experience in the development of mathematical meaning leads into a final discussion of how linguistic devices such as metonymy, metaphor, simile, and analogy can be viewed as cognitive vehicles for mathematical reasoning.

Mathematical Discourse

Discourse is becoming an increasing popular topic of conversation in mathematics education. In the year 2000, the *Principles and Standards for School Mathematics* document published by the National Council of Teachers of Mathematics (NCTM), suggested that two purposes for student participation in mathematical discourse are to (1) communicate to learn mathematics and (2) learn to communicate mathematically. Later, in 2007, NCTM's *Mathematics Teaching Today* document appealed to classroom discourse as the primary means by which students learn to define mathematical activity. It is interesting to note a shift in perspectives between these two documents published by the same organization. While the earlier document reflected a view of discourse from the individual cognition perspective of learning mathematics, the more recent document focuses on the social perspective of defining mathematical activity. These shifting perspectives of the NCTM documents are reflective of shifting perspectives on discourse and mathematics in the research community of mathematics educators as well as in the larger community of mathematicians, educators, policy makers, parents, and the general public. These shifting perspectives are guided by, and guide, empirical research on mathematical discourse in a zig zag of description and definition in efforts to not only define mathematical discourse as it currently exists, but as many believe it should exist, in the schools.

Defining Mathematical Discourse

Mathematical discourse has been said to differ from other forms of discourse in terms of vocabulary and word use. Researchers have characterized verbal communication in mathematics classrooms as using either mathematical or "ordinary" language (Pirie, 1998), while recognizing that mathematical discourse does not require the use of a specific mathematical vocabulary (Moschkovich, 2003). Ferrari (2004) has followed the tradition of Halliday (1978) in a functional linguistics evaluation of differences between mathematical and colloquial registers, which characterize "linguistic varieties according to use" (Ferrari, 2004, p. 387). While mathematical discourse can include a wide variety of language, mathematical registers are more closely related to the literate registers of written texts, and consequently the interpretation of mathematical language as colloquial language, or vice versa, may be the source of misunderstanding among participants in mathematical discourse (Ferrari, 2004; Zevenbergen, 2000). The use of mathematical and ordinary language simultaneously may result in further ambiguity due to differences in syntax and explicitness (Ferrari, 2004).

Zandieh and Knapp (2006) noted that mathematical language structures are often more rigid than natural language structures. For examples, cases of polysemy, where a word has multiple related meanings, occur quite frequently in ordinary language. A standard example is the word "foot," which can refer to the lower part of the human leg, or the lower part of a mountain or hill, as in "the foot of the hill," or "the foothills." In mathematics, however, it is less common for a mathematically defined term such as "derivative" to have a multitude of related definitions. Students may not be aware that every situation dealing with a mathematical "derivative" should be isomorphic in specific ways (Zandieh & Knapp, 2006). While mathematical definitions may attempt to avoid the ambiguity of polysemy, Zaskis (1999) points out that polysemy continues to exist, comparing the uses of "divisor" and "quotient" in whole number division to "divisor" and "quotient" in number theory. While a similar root may exist for mathematical forms of polysemy, these similarities are not always made explicit in instruction. One might compare, for example, how the term "tangent" may be defined differently in an introductory geometry course (a line tangent to a circle), a trigonometry course (the ratio of two sides of a right triangle), and a calculus course (a line that has the instantaneous slope of a curve at a point).

Mathematical discourse has also been defined in terms of content. Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) define mathematical discourse as discourse having to do with mathematical objects such as shapes or quantities. Such definitions are not entirely useful for identifying mathematical discourse because the definition of mathematical objects can vary greatly in different fields of mathematics. Furthermore, discourse about similar mathematical content can differ drastically in form. Thompson, Philipp, Thompson, and Boyd (1994) identified two orientations for mathematical discourse that they referred to as *calculational* and *conceptual*. Calculational discourse was associated with "an emphasis on performing procedures" and "a tendency to do calculations whenever an occasion to calculate presents itself" (p. 87). Conceptual discourse was driven by a teacher's "image of a system of ideas and ways of thinking that [the teacher] intends the students to develop" (p. 86). Although neither orientation was characterized by Thompson et al. as more or less mathematical, the benefits of a conceptual orientation were highlighted in the discussion.

Other definitions of discourse go far beyond the dimensions of terminology and topic to define discourse as a social practice (Gee, 1996; Moschkovich, 2003). Such definitions imply that, in order to define mathematical discourse, mathematics educators need to determine what mathematical discourse currently looks like in terms of the dimensions of behavior, interaction, values, and beliefs. Lampert and Cobb (2003) characterized this decision as being related to the decision of whether to view communication as a means for developing mathematical understanding or communication as the ends of mathematical instruction, noting that the former view is suggested by the *acquisitionist* view of learning while the latter agrees with the *participationist* metaphor (Sfard, 2001). A related issue is the sorting out of whether the success or failure of students is evidenced by their ability to communicate, caused by their ability to communicate, or even correlated with the ability to communicate, and whether the inability to communicate understanding might be misinterpreted as misunderstanding. These beliefs about what mathematical discourse does and should look like are strongly linked to different learning theories and philosophies of education.

Teacher Roles in Classroom Discourse

Nearly thirty years ago, researchers discovered a prevalent discourse pattern of teacher initiation, student reply, and teacher evaluation (I-R-E) in education (Mehan, 1979; see also Lobato, Clarke, & Ellis, 2005). This discourse pattern placed the teacher in the role of an authority who asked questions to which the answer was already known. The purpose of such questions was an evaluative purpose. The I-R-E pattern revealed the prevalence of behaviorism as a theory for learning, which suggests that immediate positive reinforcement for correct replies helps to develop internal bonds between specific academic questions and their correct answers (Resnick & Ford, 1981).

O'Connor and Michaels (1996) suggested an alternative role for the teacher in discourse in which the teacher *revoices* the words and ideas of the students, thereby situating those students in specific roles in discourses that may be "a vehicle for complex thinking and problem solving in groups" (p.95). Forman and Ansell (2001) suggested that teachers use revoicing in the mathematics classroom to "repeat, expand, recast, or translate student explanations for the speaker and the rest of the class" in order to "articulate presupposed information, emphasize particular aspects of the explanation, or disambiguate terminology (Forman et al., 1998; O'Connor & Michaels, 1993, 1996)" (p. 119).

The contrast between the teacher role in the I-R-E model and the teacher role in a discourse practice such as revoicing highlights two different views of mathematical learning and activity. The first view, known as the "sage on the stage" metaphor, is one in which the teacher is the authoritative source of knowledge and information. In a second metaphor, the teacher's role is to guide students in developing their own voices in mathematical discourse, collaboratively building ideas based on their own inquiry and experience; a role known as the "guide on the side" metaphor (Davis & Maher, 1997). Forman and Ansell (2001) found that, within one classroom, conflicting perspectives on mathematics could be observed in the talk of one elementary school teacher. When leading discussions of student strategies for solving multiplication problems, this teacher would respond to non-traditional student-developed strategies with delight, and would expound upon such strategies through revoicing. On the other hand, when students suggested the use of a traditional multiplication algorithm, the teacher would refrain from revoicing or even explaining the algorithm. The steps of traditional algorithms were merely allowed to be demonstrated, the teacher discounted such strategies as "confusing . . . to a lot of people," and explained that when she was a student, such standard algorithms were all that she was allowed to use. Although the teacher commented that traditional algorithms should be taught when students had a greater understanding of place value, it is interesting that her response to what she might call a lack of agency in her own experience was to discourage her own students' choice to use standard algorithms.

Student Roles in Classroom Discourse

The shift in a teacher's role from sage on a stage to guide on the side has had serious implications for the nature of teacher talk in the mathematics classroom. Even greater, however, are the implications for the mathematics student's role in discourse. In respect to the I-R-E pattern, Mehan (1979) stated, "As a result of the teacher's search for the one correct answer to her question, it is difficult to determine whether [a] child's answer stemmed from a mastery of the conceptual demands of the academic task, or stemmed from a mastery of the conversation demands of the questioning style" (p. 293). In an effort to shake students from focusing simply on teacher reactions, teachers have been encouraged to question all student replies, asking students to explain both correct and incorrect answers. Despite these efforts to encourage students to focus on the mathematics rather than the questioning style, some students continue to expect the I-R-E pattern in mathematical discourse. Bills (2000) studied politeness in student-teacher interactions and suggested that, although teachers may ask sincere questions or questions designed to help the student to examine their thinking and deepen their understanding, some students react to such questions as politely masked signals of their incorrect thinking. Implicit conventions of discourse such as "the teacher only interrupts when I've done something wrong" may be considered *meta-discursive rules* (Sfard, 2001), or unspoken social rules that govern mathematical discourse in many classrooms.

Conversational maxims, such as the assumption that given information is sufficient or relevant, may also be considered examples of discursive meta-rules that reflect each participant's role in the social context. As verbal mathematical discourse cannot be separated from the social context, the goal of mathematical discourse cannot be viewed as simply a negotiation of meaning or usage. Learners' relationships to each other and their ideas about mathematics and learning will also affect the way in which language is used. For example, a student may expect his instructor to have a deep understanding of mathematical concepts that the student is describing, and therefore accept vague definitions of mathematical concepts given by the instructor as sufficient (Ferrari, 2004). In the presence of such a wide variety of unspoken meta-rules with social and mathematical purposes, it is no wonder that students may misunderstand their instructors' intentions in engaging them in mathematical discourse. At an even greater disadvantage may be the students who encounter drastically different discourse patterns, with different unspoken rules, at home (Zevenbergen, 2000). Ben-Yehuda et al. (2005) analyzed differences in the arithmetical discourse of two girls with learning difficulties and suggested that students' capacities for mathematical reasoning may be unrealized or undeveloped due to these students' failure to appropriate endorsed forms of mathematical discourse.

A New Definition of Discourse and Learning

In 1990, Lampert used the writings of Lakatos (1976) and Polya (1954) to reintroduce the value of the moral qualities of courage and modesty in mathematical discourse. Lampert and her elementary school students provided an existence proof for new teacher-student interactions that involved students in the mathematical activities of knowing, thinking, revising, and explaining in a classroom where the legitimacy of mathematical ideas was determined by reasoning and mathematical argument. Along with this new view of classroom discourse came a realization that teaching students how to participate in mathematical discourse involved teaching them new social behaviors. Cobb, Wood, and Yackel (1993) recognized that two kinds of talk could be observed in a second grade mathematics classroom, *talking about mathematics*, in which students verbalized and evaluated their own interpretations of, reasoning about, and solution processes for mathematical problems, and *talking about talking about mathematics*, which consisted in explicit instruction and commentary on the former type of talk. The socially negotiated meta-rules of talking about mathematics were explicated in talking about talking about mathematics. These studies (Cobb, Wood, & Yackel, 1993; Lampert,

1990) suggested that student roles in mathematical discourse do not have to be implicitly taught or learned; rather, they can and should be explicitly suggested, modeled, and negotiated by teachers and students alike.

Although Lampert (1990) appeared to embrace what would eventually be called the *participationist* view of learning by Sfard (2001), Lampert also recognized that evidence that elementary school students were capable of participation in such mathematical discourse may not be sufficient for those interested in measuring knowledge *acquired* by her students. Nevertheless, a new type of discourse and new possibilities were up for consideration. Having broken free from the traditional constraints of I-R-E patterns, this new discourse has been much more rich and interesting for researchers to study. Even the smallest details, such as students' claims that a trapezoid is "half of a parallelogram" (Moschkovich, 2003, p. 329) are no longer framed as misconceptions to be remedied, but powerful commentaries on students' ability to notice properties, generalize, and participate in a process of linguistic invention as they operate on language to create meaning (Brown, 2001).

Pronouns, Power, and Politeness

The use of pronouns in mathematical discourse has provided powerful commentaries on beliefs about mathematics and power relationships in mathematics classrooms. Like definitions of mathematical discourse, characterizations of these beliefs and relationships have also evolved as research and practice have mutually contributed to one another. For example, the figurative use of "we" by a teacher has been identified as associating with a powerful group, or feigning solidarity (Pimm, 1987; Rowland, 1999). Ju and Kwon (2007) have suggested an alternative interpretation for a teacher's use of "we" in modern classrooms as referring in a literal sense to the students and teachers in the classroom, therefore generating a sense of "authorship and ownership" among the community of learners for the ideas that they had collaboratively developed. While teachers may refer to students as "you" in the literal sense, Rowland (1999) observed that students rarely refer to teachers in a reciprocal manner.

The figurative use of "you" as a replacement for the general "one" has been associated with generalization (Rowland, 1999). More currently, this figurative use of "you" by a student has been interpreted as a commentary on mathematics as an accessible practice, while the third person "it" and "that" obscure the role of agency in mathematics (Morgan, 2006). Wagner (2007) also associated the use of "I" with the concept of agency. Inspired by Fairclough's (1992) notion of *critical language awareness*, Wagner (2007) led discussions in an 11th grade mathematics classroom regarding the use of language in mathematics textbooks and explanations. The 11th grade participants recognized a general use of "you" and "we" as "an attempt to bridge a diversity of perspectives" (p. 42), which may be a more positively framed interpretation of the findings of Pimm and Rowland on the use of "we," or a reflection of how mathematical discourse has changed in comparison to those earlier findings.

Continuing in the tradition of critical discourse analysis, researchers have taken a closer look at discourse practices that may deny students' access to discourse. Motivated by student comments on mathematical discourse, Wagner and Herbal-Eisenmann (2008) focus on relationships between discourse particles such as the word "just" and the nature of subsequent dialogue in mathematics classrooms. Studies of classroom discourse have focused on sociolinguistic factors such as markers of politeness as well as process and continuity. For example, the phrase, “you know,” at the end of an explanation may indicate positive politeness between students to establish common ground, while an utterance preceded by, “I think,” may have the opposite effect, therefore distinguishing one’s thoughts from those that have already been expressed. In undergraduate student-led discussions, “I think,” has been identified as a marker of continuity, which allows a participant to make a comment that may not relate directly to the previous comment, yet contributes or responds to an overarching topic of discussion (Craig & Sanusi, 2003).

Additional Social Factors and Implications

In 1987, Kagan hypothesized that there may be a link between the ability to “produce divergent responses to open-ended problems and the ability to perceive others in divergent ways” (p. 183). In order to further investigate “the social implications of higher level thinking skills,” she suggested that researchers reach out from their home field of study to collaborate with those in other fields. Researchers in mathematics education have built extensively on the work of the psychologists Piaget and Vygotsky in the development of learning theories. As discussed earlier, Piaget (1997/1896) and Vygotsky (1986/1934) also studied different forms of speech. Mathematics educators have applied the notions of private and social speech in analysis of the mathematical discourse of young children (Alexander, White, & Daugherty, 1997).

Even when their intent is to study other topics such as individual cognition, mathematics educators are having a difficult time ignoring the social aspects of mathematical discourse (Cobb, Wood, & Yackel, 1993). Jirotkova and Littler (2003) found that two students participating in a communication task involving verbal descriptions of geometric solids differed in their social tendencies to build models of each other's thinking. One student would ask the other questions intended to determine how that student was using the terms "square" and "rectangle," and upon determining that her co-participant's meaning for such terms was different than her own, adapted her own language accordingly in order to collaborate effectively. This same student, however, rephrased the language and questions of the researcher with more precise language, in a possible demonstration of Goffman's (1981) notion of footing. The other student made no such attempts to modify his language. Although both students had comparable scores in their mathematics class, this exercise in verbal communication revealed drastic differences in both language and social competence.

Another linguistic notion that has gained importance in the analysis of mathematical discourse is Bakhtin's (1981) notion of dialogic discourses. After Bakhtin, Lewis and Ketter (2004) define a dialogic conversation as "one in which there is an awareness of other utterances and social meanings" (p.118). Lewis and Ketter apply their definition to a group of practicing teachers in a study group for the teaching of multi-cultural literature in a rural middle school. Their view of learning as "appropriation and reconstruction of one's social world" implies that the echo of one participant's social view in the language of another participant may be a powerful indicator of generative activity (p. 140). In concluding this section, I note that the presence of dialogism in mathematical discourse may be a key element for defining productive mathematical discourse in the future.

The Role of Personal Experience in Mathematics Learning

The use of personal experience in graph interpretation and reasoning about rates of change is the first topic of the literature reviewed here. Reform efforts in mathematics education have attempted to replicate the authentic activity of graph interpretation in the mathematics classroom, emphasizing the "need to move beyond plotting and reading points to interpreting the global meaning of a graph and the functional relationship that it describes" (Dugdale, 1993, p. 101). Graphical representations have also played a major role in calculus instruction, specifically for purposes of demonstrating the relationship between functions and their derivatives (Zandieh, 2000).

A second, related topic of this section is the study of mathematics, particularly calculus, in the context of kinematics. The relationship between displacement and velocity has become a prototypical context for the investigation of the concept of derivative (Marrongelle, 2004; Zandieh, 2000). The context of the task in this study is not one of velocity and displacement, but the rate of flow of water and quantity of water in a reservoir. Nevertheless, I review literature related to student thinking about the velocity-displacement relationship for three reasons. First, I consider the implications of drawing on personal experience to develop meaning for mathematical concepts to be relevant to my theoretical perspective of personal agency and meaning, specifically the agentive explanatory factor of experience and imagination. Second, previous to their work on the rate of flow and volume task, the participants in thus study spent 10 class sessions (approximately 16.5 hours) working on two tasks that are set in the context of displacement and velocity. Literature relevant to these two tasks, "The Desert Motion Task" (diSessa, Hammer, Sherin, & Kolpakowski, 1991), and "The Cat Task" (Speiser & Walter, 1994, 1996; Speiser, Walter, & Maher, 2003) is discussed below. Third, the participants in this study used the language of "velocity" and "displacement" as they participated in analogical reasoning about their given task. I conclude this section with an abbreviated review of the wealth of literature on metaphor, metonymy, and analogy pertinent to mathematics education in general and to this study in particular.

Graph Interpretation

Studies of graph interpretation have suggested that personal experience with the situation represented by graphs may either help or hinder students' interpretations (Johnson, 2005; Leinhardt, Zaslavsky, & Stein, 1990; Roth, 2002). Students in the middle grades were found to exhibit "iconic interpretation," or the interpretation of a graph as a picture of an event (Leinhardt et al., 1990). For example, students creating representations of a cyclist riding up and down a hill have suggested that the graph of speed versus time may be incorrect, because decreasing speed followed by increasing speed has the appearance of the shape of a valley between two hills, implying that the cyclist rode down a hill, and then up another hill (diSessa et al., 1991). Dugdale (1993) reviewed ways in which technology can create graphs of student action, resulting in students having immediate prior personal experience with the interpretation of the graph (see also Nemirovsky, 1994). These graph-creating technologies can also allow students to "test out" their interpretations of a given graph by acting out (and creating a graph of) the story that they told for the given graph. Roth (2002) reported that working professionals outperformed teaching professionals when asked to "read" a graph describing an event in their field. Roth suggested that the working professionals' performance was better because they were more familiar with the phenomena that the graphs described.

Ochs, Gonzales, and Jacoby (1996) described how members of a physics research group would incorporate the conventions of the graphical representations of physical phenomena into their gestures as they reasoned about such phenomena. For example, a physicist would accompany the phrase “come down [in temperature]” with a hand moving from right to left (p. 356). In the physical world, “down” is often associated with a falling vertical motion. But within the semiotic frame (Goodwin, 2000) of graphical representations, the direction of “down” may be determined by a vertical or horizontal axis. The physicists in the study were using a graph that represented temperature on the horizontal axis, and incorporated this convention into their discourse. The physicists also exhibited a discourse pattern of personal pronominal subjects combined with predicates of motion or change of state, as in the utterance, “When I come down I'm in the domain state” (p. 331). Ochs et al. suggested that such language functioned as a linguistic device that allowed the physicists to “symbolically participate in events” (p. 348) as they thought through problems together.

Walter and Johnson (2007) investigated the language of practicing elementary school teachers as they interpreted a graph of the rate of water entering a reservoir with respect to time. These participants spontaneously resituated the problem in the context of water entering and exiting a bathtub, and participated in a process of linguistic invention to create a story that would explain the given graph. The created story about a bathtub, along with abstract conventional language, served as a semantic warrant for the teachers’ claims about the volume, or level of water in the bathtub. Walter and Johnson defined linguistic invention as the linguistic process of relating mathematical concepts to personal experience. They defined semantic warrants as “personally meaningful, intuitive instantiations of mathematical concepts or examples to ground and reason from in building formal inferences” (p. 709; see also Weber & Alcock, 2004).

Velocity and Displacement in the Mathematics Classroom

There is not a lot of research available regarding students' interpretations of graphs of water flow in mathematical contexts (Dugdale, 1993; Gerson & Walter, 2008; Walter & Johnson, 2007). However, many authors (diSessa et al., 1991; Marrongelle, 2004; Nemirovsky, 1994; Sherin, 2000; Speiser & Walter, 1994, 1996; Speiser et al., 2003; Zandieh, 2000; Zandieh & Knapp, 2006) have researched students' interpretations and representations of motion, specifically relationships between distance, time, and velocity. Zandieh (2000) suggested that the widespread use of velocity as an instructional context for derivative may be due to the highly developed natural language used to express the ideas of displacement, velocity, and even acceleration. As experience with motion is common, students as young as 6th grade have demonstrated the potential of addressing sophisticated ideas about the relationships between time, displacement, and velocity (diSessa et al., 1991). Set in the context of inventing representations for verbal descriptions of motion, these students concluded that each of the three aspects generally used to describe motion, (speed, distance, and time) can be derived from information about the other two aspects. Therefore, a representation independently showing speed, distance, and time, was considered redundant by these young students. DiSessa et al. also observed that students preferred graphs of speed versus time to distance versus time, suggesting a natural inclination to treat speed as a primary quantity, rather than a rate of change of the primary quantity of distance. This preference of speed to distance was also reported by Nemirovsky (1994) in his paper about an 11th grade student who, with the aid of computer-based motion detector, created and interpreted graphs representing the motion of a toy car.

In their work on “The Cat Task,” Speiser and Walter (1994, 1996) and Speiser et al. (2003) suggested that the implications of basing calculus instruction in authentic data may be substantial. University faculty and high school students were given a sequence of time-lapse photographs of a cat in motion (Muybridge, 1985/1885) and were asked to draw conclusions about the motion of a cat at given points in time corresponding to given photographs in the sequence. The role of students’ personal experiences with motion, both embodied and observed, in comprehending the relationships between position, velocity, and acceleration, was examined as a vehicle for sophisticated mathematical activity. Motivated by the comments of the various participants, these authors challenged one of the major traditions of calculus instruction, that of assuming continuity of functions.

The assumption of continuity of functions in traditional instructional approaches for the concept of derivative is not a minor one. The 6^{th} grade students in diSessa et al. (1991) criticized discrete representations as not showing “what’s between the lines” (p. 137). Nemirovsky (1994) identified additional tensions between experienced reality and the conventions of mathematical representations. The notion of negative velocity on a graph is problematic because, unlike other contexts involving rate of change (inflow and outflow, for example), there is no “naturally” positive direction for displacement. Furthermore, representing negative velocity on a velocity versus time graph results in the complication of interpreting decreasing velocity in the upper half-plane as decreasing speed, but decreasing velocity in the lower half-plane as increasing speed. The idea of negative velocity, combined with the assumption of continuity, suggests the existence of a point of zero velocity when one-dimensional motion changes direction. However, the

6th graders in diSessa et al. suggested that “instantaneous stopping” was a contradiction of terms (p. 146).

Metonymy, Metaphor, and Analogy

Zandieh (2000) describes four contexts in which the concept of derivative is commonly represented. These contexts are (1) graphically as a slope, (2) verbally as a rate of change, (3) physically as velocity, and (4) symbolically as the limit of the difference quotient. Recognizing Zandieh’s list as not exhaustive, but a possible framework for evaluating student understanding, Zandieh and Knapp (2006) observed the roles of metonymy and metaphor in student reasoning about the derivative. For example, a student’s statement that “the derivative is the velocity” or “the derivative is the slope” would be considered paradigmatic metonymy, because one context of Zandieh’s framework is taken to stand for the entire concept of derivative. Presmeg (1992, 1997) recognized the widespread use of paradigmatic metonymy in mathematical statements such as “let *a* be any number,” where a single member of a set is used to represent a whole set.

Following established literary theory, Zandieh and Knapp (2006) view metaphor as differing from metonymy in that metaphor compares entities from two different conceptual domains, while metonymy compares two entities from the same conceptual domain. Therefore, a student speaking of a derivative as velocity would be using metonymy, because velocity is an example of a derivative. On the other hand, a student using their knowledge of derivative as velocity to reason about the instantaneous rate of change of temperature would be applying metaphorical reasoning because they are comparing two different contexts where one context cannot be considered a sub-context of the other.

Presmeg (1997, 1992) defined metaphor, metonymies, and similes as different forms of analogy. Metaphor and simile differ in their explicitness; a metaphor states that "A is B," while a simile states that "A is like B." In both cases, A is not the same entity as B, but the comparison is implicit in the case of metaphor, and explicit in the case of simile. Presmeg also characterized metonymy as pertaining to symbolism and metaphor as pertaining to meaning. Metonymies are primarily concerned with the representation of a class by way of one or a small collection of key members of that class. Metaphors on the other hand, allow one to reason about one class (the target) by referring to knowledge of a different class (the source). As demonstrated by Zandieh and Knapp (2006), a metaphor can also operate within a class, where one member of a class (velocity as a derivative) can be used to reason about another member of that class (the rate of change of temperature as a derivative).

Although Presmeg (1997) viewed metaphor as one type of analogy, Sfard (1997) suggested that metaphors be viewed as distinct from analogies in a specific way. While analogy may be used to reason about relationships between two extant domains or contexts, Sfard suggested that the term "metaphor" should be reserved for the specific act of creating a new domain by projecting the characteristics of previously constructed domain onto observed phenomena. For example, Sfard explains the emergence of negative numbers as the metaphorical projection of the existing positive numbers onto the set of symbols that included "impossible subtractions . . . such as 3-8 or 0-2" (p. 345). Sfard suggest that analogies are for reasoning, and metaphors are for conceptualizing. In the context of derivatives, I suggest that Sfard would consider a student's projection of the qualities of velocity onto the known, but slightly less familiar, context of the rate of change of temperature a case of analogical reasoning, while a student's projection of the qualities of velocity to conceptualize the (new to the student) idea of rate of change of temperature would be a case of metaphorical construction. Therefore, the distinction between a student's use of metaphor and a student's use of analogy would be entirely dependent upon that student's prior exposure to, or reification of, the target domain.

For the purposes of this study, I use the term *analogy* to refer to the identification of similar qualities, properties, or internal relationships of two different domains, and *analogical reasoning* to refer to all observed cases of the linguistic projection of the characteristics of one domain onto another domain. When relevant, I suggest that the specific forms of metaphor, metonymy, and simile may be present in the data, and reference the respective definitions in which I may base my claims. As I continue to view metaphor, metonymy, and simile as different forms of analogy, I conclude this section with a short review of literature on analogical problem solving.

Alexander, Willson, White, and Fuqua (1987), developed a *Test of Analogical Reasoning in Children* around four performance components for completing classical analogy problems of the form A:B::C:?. These components are encoding, inferring, mapping, and applying. Encoding refers to identification of the given terms A, B, and C. Inferring is the process of identifying relationships between A and B in the source domain. Mapping is the step of identifying a connection or similarity between A in the source domain and C in the target domain. Finally, applying refers to the appropriate application of the relationship A:B to identify an entity D, represented by the question mark, within the target domain so that C and D exhibit a relationship C:D.

Gholson, Smither, Buhrman, Duncan, and Price (1997), in a review of literature on analogical problem solving, or the process of solving a new problem by mapping it to a previously solved problem, concluded that four steps are generally recognized in the problem solving process. First, a solution to the original (base) problem in the source domain must be obtained. Second, correspondences must be noticed and identified between the base problem in the source domain and the new problem in the target domain. Third, the pertinent features of the base problem and solution must be recognized and retrieved. Fourth, these features must be mapped to the target domain, and the solution carried out. The four steps seem to include the components identified by Alexander et al. (1987) in a broader frame that allows for the solutions of more complex problems.

Gholson et al. (1997), however, remarked that an additional step is often required in the practical application of problem solving. Due to the fact that the pertinent features of different problems are not always isomorphic, analogical problem solving pragmatically involves a step of modification. A learner may attempt to modify and resolve the base problem in the source domain, make modifications during the process of mapping from the source domain to the target domain, or modify the solution process within the target domain after the process of mapping. The failure of analogical problem solving may be explained in terms of failure to complete one or more of the four original steps, or a failure to notice the need for or make appropriate modifications.

English (1997) proposed, as a direction for future research, the investigation of students' natural inclinations to use analogical linguistic forms as vehicles for mathematical reasoning. This study responds to English's proposition as it describes data in which students use analogical problem solving methods and analogical language without prior instruction. This study also focuses on a relatively new context, that of rate of flow and volume of water, for the development of the concept of derivative, and how students spontaneously connect this new context to the more widely studied context of velocity and displacement, as well as other relevant phenomena in their personal experience.

Along with describing the role of personal experience as an explanatory factor for choices made in mathematical discourse, I also describe how the additional explanatory factors of one's meaning for social roles and responsibilities, and one's concern for their own understanding of the mathematics are also reflected in the process of negotiating mathematical meaning and language. In doing so, I suggest new ways of characterizing mathematical language and social speech. Finally, this study contributes to the ongoing process of defining mathematical discourse by suggesting how evidence of the exercise of personal agency (Walter & Gerson, 2007) might serve as a criterion for defining mathematical discourse.

CHAPTER 4: METHOD

In this section, I describe the setting, participants, task, and data collection procedures for this study. I then explain how I combined grounded theory techniques with other qualitative methods as I used my data to develop two sets of codes for characterizing the process of negotiation of meaning in terms of my continua of conventional and mathematical language and egocentric and social speech. The development of these codes then led to the development of more complex codes based on emergent phenomena in the data.

Setting

This study takes place in a university honors introductory calculus classroom at a large private university in the Rocky Mountain Region of the United States. About 20 students met with two professors three mornings a week for two hours throughout the Fall Semester of 2006. Students sat at hexagonal tables in groups of four or five. Learning was task-based and investigative, and the students spent the majority of their time in class participating in small group and whole class working discussions about challenging mathematics problems and related mathematical concepts. A working discussion is characterized by progress toward shaping a solution for a designated task interlaced with discussion of goals, definitions, and implications of specific plans of action toward a solution process.

Participants

The research participants are four university students enrolled in the calculus course that sat and worked together as a group, and one of two co-instructors for the course. At the time of the course, Daniel was a sophomore majoring in actuarial science who took a statistics course and Advanced Placement calculus courses in high school. Jamie was a senior majoring in Geology who had previously taken a first-semester calculus course from another instructor at the same university. Julie was in her first semester at the university and was planning to major in mathematics education. Justin was a junior who had not declared a course of study but expressed interest in engineering, mathematics, and mathematics education. Dr. Walter was an assistant professor at the university who taught secondary mathematics for 13 years in public schools before completing her doctoral work and joining the university faculty.

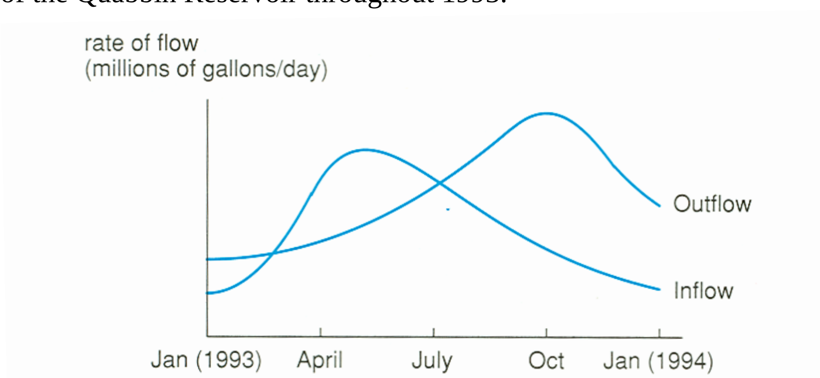
After a month of working together, the participants appeared to work comfortably with one another, as well as in the presence of a video camera. In their discussions, the students posed and answered questions about the current task, and often developed and recounted verbal explanations with the help of graphs and other inscriptions. The participants were careful not to interrupt one another and would pause in their explanations to check for understanding and agreement by other members in the group. The mood was never very heavy, and the students would joke with one another and laugh at intervals throughout their discussions.

The Quabbin Reservoir Task

On October 18, 2006, the participants had a working discussion of the Quabbin Reservoir Task (Figure 3), which the instructors adapted from Hughes-Hallett et al. (1994). In the task, the participants are given graphs of inflow and outflow of water in a reservoir over the period of one year, and asked describe and create graphs of the volume of water in the reservoir for that year.

It should be noted that, prior to the instructors' decision to adapt the Quabbin Reservoir Task for use in the calculus class described here, the Quabbin Reservoir graph originally appeared at another university as part of a calculus assessment item in which students were asked to justify whether there was more water in the reservoir in January 1993 or January 1994. Hughes-Hallett et al. incorporated the task into a Calculus textbook as an exercise, asking slightly different questions. In this study, the instructors of the calculus course adapted the Quabbin Reservoir Task based on the previous work and emergent ideas of the students in the course.

The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph below represents the flow of water in and out of the Quabbin Reservoir throughout 1993.



a. Sketch a possible graph for the quantity of water in the reservoir, as a function of time.
b. Explain the changes in the quantity of water in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year. How are these changes evident in your graph in part (a)?

Figure 3. Quabbin Reservoir Task.

The Quabbin Reservoir Task was given to the students in the calculus course on a sheet of paper as shown in Appendix A. The students had no previous experience with the Quabbin Reservoir Task, and were given no additional instruction beyond the instructions printed on the task sheet. The instructors' purpose in using the task in this course was not as an assessment or exercise, but for developing students' conceptual understandings of the derivative in a new context. The instructors also anticipated that the Quabbin Reservoir Task would elicit discussion of important calculus concepts such as anti-derivatives, extrema, concavity, points of inflection, and an interpretation of the area between curves. As mentioned earlier, the participants had previously spent 10 class periods working on tasks set in contexts of motion, and had interpreted velocity as the rate of change of displacement with respect to time. What became problematic for these students, then, was determining how to interpret the relationship between the given graphs of inflow and outflow and the desired graph of quantity, and connecting this relationship to their previous interpretations of the derivative as an instantaneous rate of change.

Data Collection

A team of graduate and undergraduate student researchers videotaped each classroom session. The student at the camera focused primarily on the discourse of the focus group mentioned previously, except in instances of group presentations and whole class discussions. During such presentations and discussions, the camera operator would focus on capturing the words and inscriptions of students at the white board and any questions directed toward those students. If members of the focus group engaged in mathematical conversations with each other during whole-class discussions or student presentations, the camera operator was faced with the decision of determining whether the conversation of the focus group or the activity of the larger class discussion was more relevant to the research interests of the research team. Thus, although the camera was placed in the classroom to capture as much detail as possible, the creation of video data by nature resulted in an edited version of the actual events that occurred in the classroom.

During the collection of the data presented in the next chapter, the participants' discussion was not interrupted by any whole class activity. However, there were stretches of time when the participants would hold two separate conversations simultaneously. In these instances, although the microphone could capture the overlapping talk of the two conversations, for the sake of continuity the camera operator focused on the interactions, gestures and inscriptions of just one conversation for the duration of the overlapping conversations.

As part of their regular coursework, the participants were required to submit a written summary of their solution and solution process, along with brief narratives of their developmental understandings of key mathematical ideas that emerged for them individually, in group work, or in class discussion of the task. These student write-ups served along with the video as sources of data.

Along with my presence as a student researcher in the classroom, survey responses and a follow-up interview also informed my analysis in peripheral ways. The participants completed an initial survey about their beliefs on mathematics, mathematics learning, and responsibilities of teachers and students at the beginning of this course. I present some of the responses from this initial survey at the beginning of Chapter 5 to aid the reader in developing a sense for Justin's meaning for mathematics and social roles and responsibilities in a mathematics classroom.

Most of the participants were quite vocal in the mathematical discourse, providing ample opportunities for analysis of their language use in the negotiation of meaning. Julie, however, was not as vocal (see Appendix B for information about the amount of participation in discourse) and analysis of her language left me with many questions. For this reason, I conducted a follow-up interview with Julie during the final stages of analysis. During this interview, I showed Julie a ten-minute segment of video and asked if she could elaborate on what her intent was in making a particular statement in that video. As this interview occurred nearly two years after the original data was collected, Julie admitted that she was not entirely certain of her intent. However, her statements in this interview seemed to support, and did not contradict, emergent explanations based on the video data.

Transcript Creation and Conventions

I transcribed approximately one hour of video from the first full day of the participants' work on the Quabbin Reservoir Task. As the participants often created and referred to their inscriptions as part of their discussion, I used photocopies of these inscriptions to help clarify and annotate utterances in the transcript. Research team members verified the transcript.

In creating and annotating the transcript, the following conventions were used. Interrupted speech is notated by a hyphen (-) at the point of the interruption. The next line of transcript indicates the interrupting portion. If the interrupted speech is continued, the next line with the same speaker begins with a hyphen (-) to indicate this continuation. Gestures are shown in normal font in square brackets. My personal interpretations as to the referents of pronouns used in discourse are also in square brackets, but in *italicized* text. Pauses in speech are also notated in square brackets by the number of seconds of the pause, for example: [2 sec].

For navigation purposes, I labeled references in the transcript to specific intervals on the graph according to letter labels developed by Daniel late in the transcript. As shown in Figure 4, Daniel labeled January 1993 as point A; B was the first “zero point,” at the first intersection of the inflow and outflow; C was at April; D was approximately at the maximum of the inflow; E was at either July or the second zero point; F was at October; and G was at January 1994.

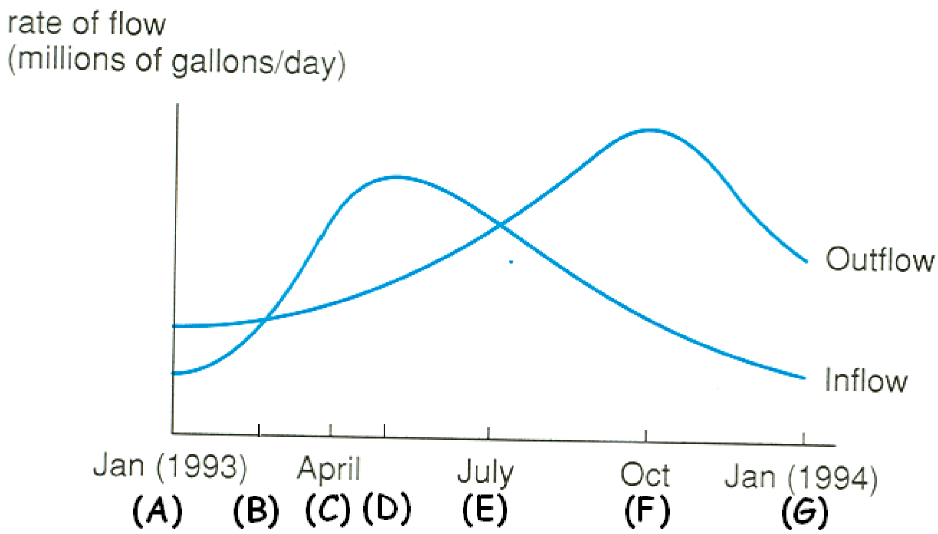


Figure 4. Daniel’s letter labels for the horizontal axis.

I originally separated the transcript into turns, or segments of uninterrupted speech by a speaker. Exceptionally long turns (greater than 30 seconds) were broken into smaller segments at natural pauses in speech or changes in topic. For example, Turn 58 (below) is an uninterrupted segment of Daniel’s speech that describes related ideas.

58 Daniel: So, it has a negative slope. And then it starts going positive up to that point [point E, July, on the rate graph]. And so it levels off at zero [point E, July, on the volume graph]. Cause the v-, the v- [1 sec] I don’t know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. [2 sec] Which is correct on our velocity chart. And then it starts going negative again. And it starts, kind of, sloping out. And it has, its greatest slope is right here [point F, October, on the original graph], so that’s its inflection point [point F, October, on the volume graph].

However, as eventual coding and data analysis focused more closely on individual words and phrases, the length of this turn obscured the detail of the coding. Therefore, I broke this turn into two parts, 58a and 58b. The choice of where to split the turn was based on the natural change in Daniel's attention to different intervals on the graph when he finishes describing the point where the "velocity of the flow of the water is zero," and then begins describing the next interval on the graph, where "[the velocity] starts going negative again." The majority of long turns occurred in similar situations in which the participants were describing the shapes of various graphs or narrating the changes in the graph with respect to time. Verbal cues such as Daniel's "and then" helped me to determine when narrative speech was moving forward to describe the next interval of the graph and I used these verbal cues to break turns into smaller portions when necessary.

58a Daniel: So, it has a negative slope. And then it starts going positive up to that point [point E, July, on the rate graph]. And so it levels off at zero [point E, July, on the volume graph]. Cause the v-, the v- [1 sec] I don't know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. [2 sec] Which is correct on our velocity chart.

58b Daniel: And then it starts going negative again. And it starts, kind of, sloping out. And it has, its greatest slope is right here [point F, October, on the original graph], so that's its inflection point [point F, October, on the volume graph].

I used Transana software (Fassnacht & Woods, 2008) to link each of these shorter turns in transcript to corresponding video clips. When all of the turns were 30 seconds or less, I named these new segments of transcript "clips," both to be consistent with the convention of Transana, and also as a way of reminding myself of the importance of using transcript to aid my analysis of my primary source of video data, rather than demoting the video to the secondary position of merely supporting my analysis of the written transcript.

Data Analysis

I combined grounded theory methodology (Strauss & Corbin, 1998) with my original interests in social and egocentric speech and conventional and ordinary mathematical language to conduct a constant comparative analysis of the data. Grounded theory methodology implies the use of codes to interpret data. Open codes were initially used to identify and delineate categories of language use and events in mathematical discourse. As subcategories developed within open codes, I identified relationships between different subcategories, which led to the development of new codes and coding cycles. As I continuously reviewed my data during these coding cycles, I identified patterns and processes in the negotiation of mathematical meaning. I also focused in on codes that I felt were most difficult to explain, viewing apparent contradictions in coding and data as opportunities for me to take a different or closer look at the data. In the following section, I describe two initial sets of open codes that were based on my theoretical perspective.

Pronoun Codes

As I was interested in the role of egocentric and social speech in mathematical discourse, I initially attempted to code the various clips as involving either egocentric or social speech. However, I soon found this coding process quite arbitrary and subjective as it was difficult to determine whether a speaker was attempting to “place himself at the point of view of the hearer” (Piaget, 1997/1896, p. 9). To further confound matters, the participants in my study were so attentive to one another that even if a comment was made with no designated “hearer,” the other participants treated the comment as if it were social speech and, “hearing” each other, would attempt to interpret or respond to the comment. For example, Daniel's coining of the phrase "concavitivi-ness-es" appeared to be a verbal accompaniment of a personal attempt to label his graph, or an instance of "thinking out loud." However, Jamie, upon hearing Daniel's supposedly egocentric speech commented "Good word!" which was responded to by Daniel with, "I like to make up words." I reasoned that if I was aware of the participants' attentiveness to one another, they were probably also aware of the fact that virtually nothing they ever said would go unnoticed by the other participants. Justin's singing, "Inflection, flection, what's your-" seemed related to Piaget's repetitive "echolalia" (p. 12) category of egocentric speech, but when Jamie and Daniel chimed in "-flection," they revealed that, even if Justin had intended to sing for his own benefit, he definitely had an audience. On the other hand, Julie would often make quiet statements that could have been interpreted as her personal summary of the current discussion, or a request for verification of her summary of the current discussion. Piaget had identified social and egocentric speech in terms of the intent of the speaker, and at this point in my analysis I did not feel as though I was familiar enough with my data to draw conclusions about the intents of my participants. For this reason, I abandoned the notion of objectively coding social speech and egocentric speech. In Chapters 6 and 7, I describe how other emergent codes revealed ways in which these participants chose language that reflected the point of view of the hearer, suggesting that these participants used social speech in specific ways in the process of negotiation of meaning.

One thing that I felt could be coded a little more objectively in this initial process of getting to know my data was the actual words used in clips of discourse. I anticipated that first, second, and third person pronoun use might be indicative of types of speech related to certain varieties of social and egocentric speech. For example, the literal use of you, as in “you worked backwards [to solve the problem]” might serve as strong evidence of the speaker taking on the point of view of the hearer. Therefore, my first set of codes described the use of pronouns in each clip. (A list of pronoun codes and the number of clips in which they occurred can be found in Appendix C).

Impersonal pronouns with a known or unknown referent were coded according to their form. Personal pronouns, specifically “I” and “you” were coded according to their referent. If “I” was used to refer to the speaker, this was coded as “I-personal.” If “I” was used to express the imagined speech of a mathematical object, this was coded as “I-personifying.” There were similarly two types of “they;” “they-inanimate” referred to mathematical objects as in “where they [the lines on the graph] meet,” and “they” referred to known or unknown human beings as in “they gave us this graph.” Four codes for “you” emerged from the data. “You-the hearer” was the most common form of you, as in “Will you explain it to me?” When “you-the hearer” was directed at multiple hearers, as in “you guys,” these instances were also coded as “you-plural.” Another form of “you” was present in the data, which did not seem to refer to any particular participant in the discussion, but rather appeared to function as the word “one” functions in “this is what one does.” Although the participants never used “one” in this manner in the data, there were many instances of “you” in the same function, as in “this is what you do.” Clips including this general use of “you” were coded as “you-one.” Finally, there was a form of “you-personified” that paralleled “I-personified,” in that it functioned as if one were speaking to a mathematical object. These personal pronoun codes were also applied to their possessive adjective forms (my, your, our, their, her, his). These forms were reflected in the codes by the additional assignment of the “possessive” code to clips of discourse in which they occurred.

Vocabulary Codes

As I was also interested in the role of conventional and ordinary language in the negotiation of mathematical meaning, I coded significant vocabulary words that were used to describe the mathematical concepts and process in the discourse. In nearly 900 clips of video data, 183 vocabulary codes emerged. A list of these codes and their frequencies can be seen in Appendix D. These codes were developed based on specific words in their various grammatical forms and functions. For example, one code, “flow,” was assigned to both noun and verb functions as well as the various verb forms such as “flowing.” Individual codes were developed for words that were unique to the data, such as “positivity.” Although there already were codes for “positive,” and “negative,” I considered Daniel’s unique use of “positivity,” and “negativity,” to be important in the development of mathematical language and meaning through discourse.

In the next chapter I describe how I built upon these two initial sets of codes to create new sets of codes that reflected the emerging phenomena in the data, and then used all of my codes to construct narratives of the negotiation of meaning.

CHAPTER 5: DATA AND ANALYSIS

I used grounded theory techniques to build upon my pronoun codes and vocabulary codes to develop additional sets of codes. In this chapter, I describe the development of concept codes, which may be considered axial codes in that they help to identify relationships between the various vocabulary codes. I also describe the emergence of a set of language awareness codes that reflected the participants' subtle verbal cues and more explicit acknowledgements of their participation in processes of negotiation. I finish this chapter by briefly describing how the presence of language awareness codes in an eight minute long segment of the video data that I named "The Gospel According to Justin," directed a comparative analysis of concept codes and vocabulary codes which resulted in the construction of three narratives of the negotiation of meaning. These narratives are primarily told through the language of Justin, and converge in his language choices in "The Gospel According to Justin." Because the narratives suggest how Justin's (1) experiences and imagination, (2) meaning for social roles and responsibilities, and (3) mathematical understandings may explain his choices in discourse, this section begins with a review of peripheral data about Justin that informed analysis. After presenting this peripheral data, I also orient the reader with a timeline and short descriptions of major segments of discourse.

Justin's Initial Participant Survey

On the first day of the calculus course, all of the participants were asked to complete a background information survey. This survey asked the participants what they like most about mathematics. Justin's response was, "Unlike some other fields of study, the solutions to problems in math are definite and exact. There may be an infinite number of solutions but that is itself a definite answer. In most cases, personal bias does not play a factor in math." When asked to list three qualities of an excellent mathematics learner, Justin listed, "patience," "keeping it simple," and "humility." He said that, of these three qualities, patience was the quality he considered his strength, and "keeping it simple" was the area in which he was the weakest. He wrote, "I definitely have a problem keeping things simple. I always want to make things more difficult than they need to be, which almost always complicates the problem and makes discovering the solution all that much more difficult." In the data presented here, Justin makes choices that simplify and organize the language and ideas encountered in mathematical discourse.

When asked on the initial survey what he felt were the responsibilities of a student in the course, Justin replied, "Learn, pay attention**,** teach, and not disturb the learning atmosphere of others." Justin expressed hesitancy about group work, listing it as the thing he found least appealing about mathematics. He wrote, "Unfortunately, I'm not a big fan of working in groups. But, since this class appears to work extensively in groups I'm just gonna have to deal with it and learn to like it. I have a hard time trusting the work that other people do or trusting that they will do the work asked of them. It's a bad habit, but I need to work on it so this should be a good class for me." Over the course of the semester, Justin emerged as a co-operative leader in his group. He would organize group activity, asking the group where they currently were and where they were going.

Initial Work on the Quabbin Reservoir Task

Although I have focused my analysis on student discourse from October 18, 2006, some details from the previous class period, October 16, 2006, are particularly relevant to my results. With approximately 20 minutes remaining in the 2 hour class period on

October 16, the participants were given the Quabbin Reservoir Task, with the only verbal instruction being to "spend the remainder of class time working on this and then your homework is not necessarily to finish this, but maybe think through and work on it."

Making Sense

During these initial 20 minutes of collaboration on the task, the participants' language suggests that their choices were guided by the general principle of "making sense." Daniel initially sketched a graph of volume (Figure 5), Julie sketched a graph that averaged the values of the inflow and outflow graphs (Figure 6), and Justin used his pen as a measuring tool to sketch a graph of the difference between the values of the inflow and outflow graphs (Figure 7).

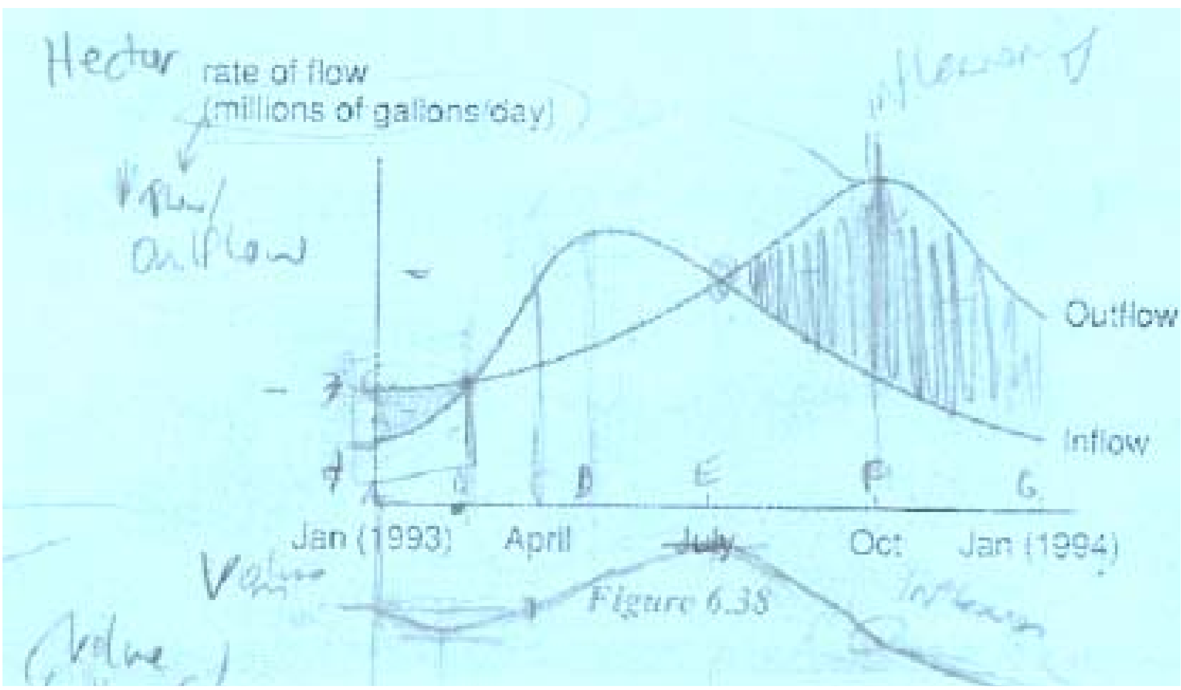


Figure 5. Daniel's graph of volume sketched underneath the original graph.

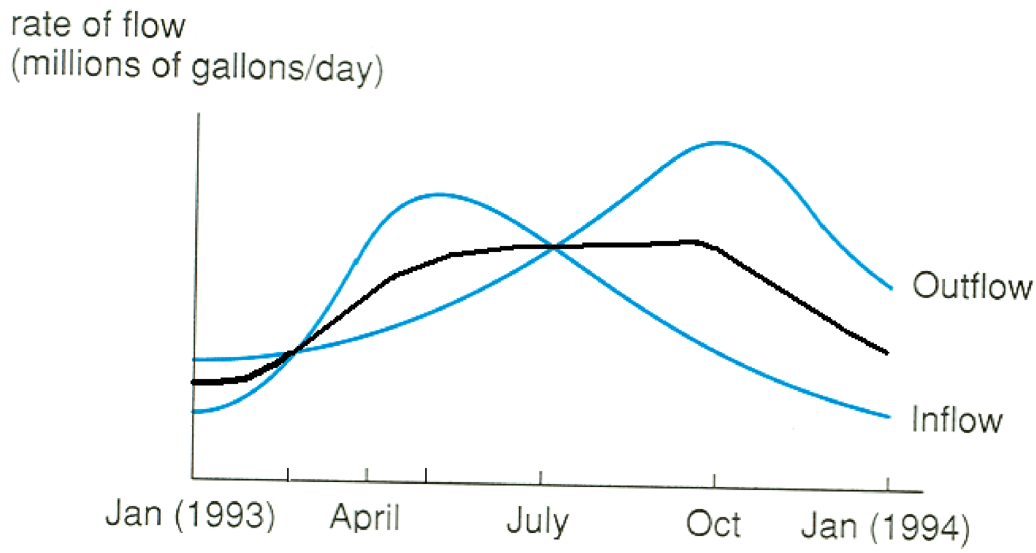


Figure 6. Julie's graph (reproduction).

In the transcript that follows, Justin explains to Andrew, a student who was absent for the next class period and therefore not a participant in the major portion of data analyzed here, that it "makes the most sense," to subtract the outflow from the inflow, rather than the other way around.

Justin: It would just depend on how you look at it. If you want to take, let's see. You would take the inflow minus the outflow. You have to decide if you want it to be the inflow minus the outflow or the outflow minus the inflow.
Andrew: Mm hmm.
Justin: So probably go with inflow minus outflow because that makes the most sense.
Andrew: Okay.

The majority of the discussion among the participants was toward the goal of determining how the graphs that they had constructed were related to one another. Justin conjectured that Julie's graph was a vertical translation of Daniel's graph, and that the only difference was the initial amount of volume in the reservoir. Claiming that his graph represented the change of water in the reservoir, Justin attempted interpret his graph in terms of relative amounts of water in the reservoir, but abandoned his interpretation because it didn't "make sense."

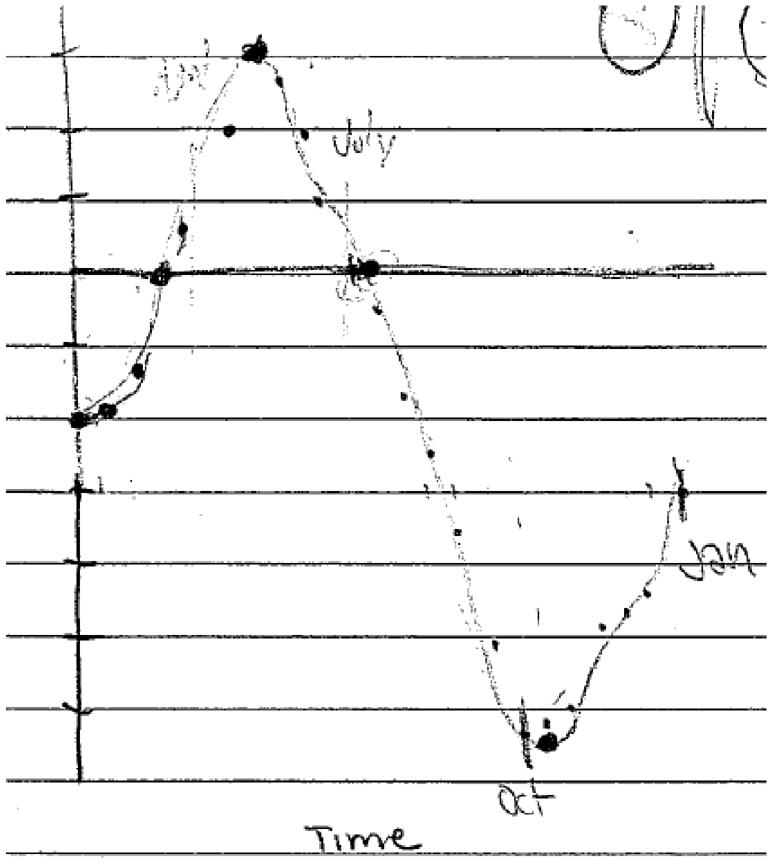


Figure 7. Justin's graph.

Justin: So that's the change in the water and so it doesn't really matter how much water's in there, that's just how much it changes. So if your starting point, this is how much water there is [horizontal axis of Justin's rate graph], after your first month it's going to be well, this point's lower than what your regular is [initial value of Justin's rate graph]. And then up here it's [first x-intercept of Justin's rate graph], wait no, that doesn't make sense. Yeah, anyways.

Shortly before the end of the class, Daniel connected Justin's graph to his own graph by comparing the signs of the values on Justin's rate graph to the signs of the slopes of Daniel's volume graph. Daniel also explained that the maximum value on his volume graph corresponded with the second x-intercept on Justin's rate graph. Because

Daniel's comparisons "made sense," Justin suggested that he "accidentally" took the derivative of Daniel's graph.

Justin: Well that would make sense. So I accidentally took the derivative of that.

As the participants left the classroom Justin confidently stated that they had "accidentally discovered the whole plan of this whole paper [the Quabbin Reservoir Task]." Although the current solution may have "made sense" to Justin, there was still much recounting, explaining, justifying, and negotiation of meaning and language that would take place before the four group members were comfortable with their solution. These processes, which primarily involved decisions about language, justification, and explanation, all occurred during the second day of work on the Quabbin Reservoir Task, which is the data that I focus on in this study.

Analogical Language

During this initial period of work on the task, Justin, Daniel, Jamie, and Andrew frequently referred to "velocity graphs" and "displacement graphs." This choice of language may be best explained by the students' shared experience working on tasks in which they had developed the concept of derivative by analyzing situations that involved velocity and displacement. Justin appeared to be particularly comfortable with this metaphorical language. When the group began to work on the task, Andrew asked about the interpretation of the graph given in the task.

Andrew: So is this, what is this graph then? Is this a displacement?
Justin: It's rate, rate of flow. So it would be-
Andrew: Displacement?
Justin: No, it would be like velocity.

It is interesting that Justin noted that the graph would be "like" velocity, using the linguistic device of simile, while Andrew spoke more metaphorically, asking if the graph was “a displacement [graph].” There were initially comments that “velocity” may not be the appropriate term. Daniel, at one point said, “I hesitate to say that. It’s not really *velocity* of the water but . . .” With time, however, comments of this nature began to diminish and the participants used the terms “velocity” when speaking of rates of change and “displacement” when speaking of graphs of quantities without reservation. The following exchange between Jamie and Daniel demonstrates how the members of the group drew on their shared mathematical experiences to create a paradigmatic metonymy (Zandieh & Knapp, 2006) of the velocity-displacement relationship to communicate about the concept of derivative.

Jamie:	But remember how displacement and velocity graphs, velocity graph was the derivative of our displacement graph.
Daniel:	Yeah.
Jamie:	So they can’t be, one can’t be, hers can’t be velocity and yours be displacement.
Daniel:	Why not?
Jamie:	‘Cause they’re the same graph. Yes? No?

In the second day of work on the Quabbin Reservoir Task, the participants negotiate language and meaning as they come to a consensus on what their various graphs represent, how they want to talk about the graphs, how the graphs are related to one another, and how the graphs represent progress toward a solution to the task.

Overview of Mathematical Discourse

Before looking at the second day in detail, I present a chronological storyline of the first hour of the second day of discourse about the Quabbin Reservoir Task. This storyline breaks the 48 minutes of video data into 14 segments which I named, when possible, after key phrases from the student discourse. The names, duration, and clips included in each segment are shown in Table 1.

Table 1
Timeline of Discourse Segments

Segment	Name	Clips	Duration
1	Getting Started	21-41	00:48.5
2	Working Backwards	42-50	00:31.0
3	Daniel and Julie	49-64	01:33.2
4	Daniel and Jamie	62-140	04:43.1
5	The Inflection Idea and the Concave Thing	141-187	02:51.3
6	Zero Points	188-233	04:02.4
7	What Are We Trying to Find?	234-283	01:53.1
8	Language for Quantity	284-329	03:50.5
9	The Gospel According to Justin	330-452	08:40.6
10	Monkey Wrench	451-555	05:22.0
11	Anti-derivative vs. Integral	556-621	03:00.6
12	Labeling	622-665	01:53.8
13	Breaking it into Quarters	666-884	08:53.2
14	Comparing Volumes	885-906	01:38.9

I now provide a brief narrative of the discourse in each segment.

Segment 1: Getting Started

The class period begins with a short survey about limits. After the surveys were collected and the students commented about how they responded to some of the items on the survey, Justin encourages his group to continue their work on the Quabbin Reservoir Task by clapping his hands together in front of him and saying, “Let’s go. So, how, what did we learn?” Jamie, Daniel, and Justin all admit that they did not work on the task since their last class. Justin moves the conversation along by summarizing the work that the group had done collectively on the task during the previous class period.

This discussion is briefly interrupted as one of the instructors approaches the table and informs the group that they will be asked to present on what they feel are “key points” of the Quabbin Reservoir Task in approximately 30 minutes.

Segment 2: Working Backwards

In this short segment, Justin briefly describes his interpretation of what happened in the previous class period. He says that he spent his time consolidating the inflow and outflow graphs to see what “the derivative will look like,” and that Daniel, Julie, and Jamie “worked backwards” to find what the water level would look like. Daniel asks if everyone understood this previous work.

Segment 3: Daniel and Julie

As a response to Daniel’s checking for understanding, Julie begins to ask Daniel questions about his approach to the task. Assuming that Julie has created a similar graph of net rate, Daniel begins to explain how he interpreted the values of the net rate graph as slopes of the volume graph. He uses the term “velocity,” speaking of a point where the “velocity is zero,” and also identifies the point with the greatest slope as an “inflection point.” During this time, Dr. Walter, one of the two professors teaching the course, comes to the table and begins to observe the discussion.

Segment 4: Daniel and Jamie

After Daniel has explained the process of creating a net rate graph and a volume graph to Julie, Jamie asks Daniel to help her understand “the labeling.” In order to explain which graph is which, Daniel recounts the solution process of consolidating the inflow and outflow graphs to create a net rate graph. Daniel then re-explains how he used the net rate graph, which he calls “the happy flow chart of our velocity” to construct a volume graph. Daniel claims that the point with the greatest outflow will be a point of inflection. Jamie asks Daniel why she can’t see the point of inflection on his volume graph. Daniel emphasizes the shape of the point of inflection as “where it just goes down and then starts to level off.” Daniel says that the inflection point is also “where the velocity is the highest.” Later, Daniel also recognizes that an inflection point can be where the velocity is the lowest, demonstrating a more complete view of inflection points of a function as extrema of the derivative of the function.

Segment 5: The Inflection Idea and the Concave Thing

Julie states that she doesn’t think that she understands the inflection idea. Using his previous idea of inflection point as “where the velocity is the highest,” Daniel invites Julie to think of riding down a playground slide and imagining where her speed would be the highest. Daniel also invites Julie to draw a curve, attempting to “feel” where the curve starts “to curve off.” Julie asks if there is a connection between inflection points and concavity. Daniel admits that he does not remember “the concave thing,” but Justin reaches across the table and identifies the concave down and concave up intervals on the curve that Daniel has been drawing.

Segment 6: Zero Points

Julie asks Daniel for some help in refining her volume graph. Daniel identifies what he calls “zero points” (conventionally referred to as critical points, or the zeros of the derivative function) as important in shaping the volume graph. Dr. Walter, one of the two instructors for the calculus course, asks Daniel to clarify what he means by “zero points.” Justin, Daniel, and eventually Jamie all voice their interpretations of “zero points.”

Segment 7: What Are We Trying to Find?

Julie states, “I think I’m still confused with the idea that it’s velocity,” sparking a discussion of alternative vocabulary for the task. This discussion quickly leads into a discussion of what the goals of the task are. Justin re-reads the instructions for part (a), "sketch a possible graph for the quantity of water in the reservoir, as a function of time," and offers his interpretation of what they have been asked to do.

Segment 8: Language for Quantity

The discussion returns to the topic of vocabulary as Dr. Walter asks the students to consider appropriate terminology for a graph of quantity of water. After considering the fact that the given inflow and outflow graphs are in terms of "millions of gallons per day," Justin concludes that the quantity graph should be in terms of volume. Recalling that there is special language for derivatives in the contexts that have been previously encountered in the calculus course, Daniel wonders whether there is a special name for the "derivative of volume."

Segment 9: The Gospel According to Justin

As Daniel begins pursuing a name for "the derivative of volume" in his textbook, Justin begins an in-depth conversation with Julie about the goals and processes of the Quabbin Reservoir Task. Justin calls his explanation, "The Gospel According to Justin." First, Justin explains the process of adding the inflow and outflow graphs together. Next he explains how the shape of the net rate graph dictates the shape of the volume graph. He describes this process of constructing a function from its derivative as "working backwards." Finally, Justin summarizes his work on the Quabbin Reservoir Task and Julie identifies the critical zero points on the rate graph as extrema on the volume graph.

Segment 10: Monkey Wrench

"The Gospel According to Justin" ends as Daniel introduces a "monkey wrench" that he has found in the calculus book. In his quest to find an official name for the "derivative of volume," Daniel has noticed that if you take the derivative of the formula for the volume of certain geometric solids, you end up with the formula for the surface area of the solid. Although this connection is only true for derivatives with respect to the radii of these solids, the participants attempt to interpret surface area as a rate of change of volume. They finally give up, partially because they can't agree on a definition for surface area, and partially because they feel as though they would have to know the shape of their reservoir in order to implement Daniel's idea in a useful way.

Segment 11: Anti-derivative vs. Integral

Dr. Walter questions Daniel's use of the term "anti-derivative," and asks if "anti-derivative" is connected to what Justin has described as "working backwards." Justin comments that he understood anti-derivatives as "integrals." Basing his description in his previous calculus experience, Daniel suggests that "integral" refers to a symbolic process of "how you find the anti-derivative."

Segment 12: Labeling

In this segment, the participants decide that the derivative graph should be labeled "rate of change in volume" and that the quantity graph should be called a "volume graph."

Segment 13: Breaking it into Quarters

The participants move on to part (b) of the Quabbin Reservoir Task, which asks them to "Explain the changes in the quantity of water in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year," relating these changes to the volume graph that they have created. The students determine that each member of the group should focus on explaining one of the four quarters of the year, since there are four quarters, and four group members.

Segment 14: Comparing Volumes

Dr. Walter asks the participants to compare the amount of water in the reservoir on January first to the amount of water in the reservoir on April first. Justin begins to interpret the area between the inflow and outflow curves on the given graph as volume.

Development of Concept Codes

After coding my data for pronouns, and vocabulary, I hypothesized that the participants were using more than one word to communicate similar concepts. I wanted to see if I could consolidate my vocabulary codes into concept categories. While reviewing the chronological occurrences of codes in discourse, I observed that the term “velocity” was used quite frequently (in 29 clips, on average 1.8 clips per minute) for the first 15 minutes of the discussion, but after that point (clip 277) the term “velocity” was used quite sparsely (in only 8 clips for the remaining 33 minutes of discussion, or .25 clips per minute). The term “volume,” on the other hand, was not used for the first 15 minutes of the discussion, but was used frequently for the remaining 33 minutes (it first appeared in clip 286 and continued to appear in an average 1.9 clips per minute for a total of 57 clips in the last 33 minutes). Although numerically, “volume” seemed to have filled a role that “velocity” may have vacated, this conclusion didn’t seem to agree with my initial reviews of the data because it didn’t seem like the two words were analogous in their use. The participants generally used velocity to talk about a rate of change of a quantity, but volume represented a quantity that wasn’t necessarily changing. It didn’t seem possible that the concept of “volume,” could have taken on the conceptual role in discourse formerly filled by "velocity." I wondered if I could make a collection of words that did fulfill certain conceptual roles. To pursue my questions about the role of "velocity," and "volume," I first tried to determine if there was a collection of words that were used to portray the same idea as volume, or more generally, quantity. The words that I initially looked at were: amount, displacement, gallons, how much, quantity, volume, water, and water level. I wanted to know if these words were being used in a similar way, so I used the search function of Transana to find clips where these words were used in conjunction with one another. I now examine some of these clips in detail to describe the development of concept codes.

Clip 891

I found that the terms "amount," "gallons," "volume,", and "water level," were all used together in clip 891. This clip occurs in Segment 14, when Justin is comparing the amount of volume in the reservoir on January first to the amount of volume in the reservoir on April first.

891 Justin: I would say, hmm, they're probably about equal, like it would be at the same, wherever you start, your **water level** at January first, or your, the **amount** of **volume** [motioning as in holding a quantity with arms wide], excuse me, the uh, millions of **gallons**, will be approximately the same, as in April.

"Amount" was used in conjunction with "volume," in the phrase "amount of volume." This amount of volume was static, as it was the amount of volume at one point in time, on January first. Justin also uses the terms "water level," and "millions of gallons," in a sequence with "amount of volume." Justin often would clarify his language by using one word and then providing a series of replacements, or what appeared to be synonyms, for the original word. At this point in the analysis, I was unsure whether he intended these to be synonyms for the sake of explanation and richer interpretation, or if he considered them to vary in precision or meaning in significant ways.

Clip 339

The search found both "amount" and "water" used in clip 339, when Justin is explaining to Julie how he consolidated the inflow and the outflow graphs.

339 Justin: So like, so you take the, you start, start with the income, uh, inflow I'm sorry, the inflow and you subtract the outflow from that part right, that's gonna give you the **amount** of **water** that's either "coming in" or "leaving," [making quotation marks in the air with his fingers] if it's negative it's leaving if it's positive it's, it's coming in.

Here Justin uses "amount" in the phrase "amount of water." Although "amount of water," may seem similar to "amount of volume," this clip was coded differently from clip 891. In clip 891, Justin used "amount of volume" to refer to the volume of water in the reservoir at a given time. In clip 339, however, Justin is not referring to the amount of water that is in the reservoir, but "the amount of water that's either 'coming in,' or 'leaving'" to describe a rate of change or how the volume of the reservoir changes. The use of "amount," in clip 339 was coded as *rate of change in volume,* while "amount" in clip 891 was coded as *volume.*

Again, in clip 339 we see Justin correct his language, replacing "income," with "inflow," in a manner that seems somewhat different from clip 891. He is apologetic, as if "income," is inappropriate or confusing for the situation. I considered this another example of Justin's language awareness, making a note that I should definitely return to investigate Justin's language substitutions. These substitutions and other verbal cues were later summarized in a set of language awareness codes (see page 78).

Clip 260

As Justin attempts to explain the goals of part (a) of the Quabbin Reservoir Task, he juxtaposes the terms "how much" and "water" in clip 260.

260 Justin: So they've given us a rate of flow graph. Which means it's telling us **how much water** is entering [pulling right hand to the right in front of body] this reservoir and **how much** it's filling up [lifting raising right hand while holding left hand steady below it], but also **how much** is leaving [pushing right hand away from self to the right] the reservoir at that exact same time, and so **how much** it's going down [lowering right hand to meet left hand held steady].

Again, although he refers to a volume of water, Justin describes the particular volume of water that represents the change in the overall volume at a given time. Clip 260 was therefore coded as *rate of change in volume.* As I reviewed instances of "how much" and "amount," I found that these terms were generally used in conjunction with "water is leaving/entering the reservoir" to describe a rate of change of volume rather than a specific measurement of volume in the reservoir.

Clip 281

In clip 281, Julie has just stated that rate of flow, or velocity, is not "what they are trying to find." Justin again rephrases his interpretation of the task.

281 Justin: No, we're trying to find, we're trying to take this graph [points to the original graph], they're telling us **how much** the **water level** is changing [moving hands together and apart vertically with palms facing] and make uh, our "best guess" [making quotation marks in the air with his fingers] at what the **water level** looks like [holding two hands with palms facing as before], a graph of how, [drops the left hand and just moves the left hand slowly up and down with palm facing downward] the **water level** change over time. Does that make sense?

Justin's reply suggests that there is a difference between a graph that "tells us how much the water level is changing" (his description of the information that was given in the problem) and "a graph of . . . the water level change over time" (his description of what they are trying to find). Both descriptions employ the terms "water level," and "change." These two descriptions may seem like they are descriptions of the same thing, and one might question why the other participants respond in the affirmative to Justin's final questions of "Does that make sense?" But closer inspection shows that the first description employs the term "how much," while the second employs "over time." I concluded that these two small terms signal a significant difference in meaning for Justin. While both may be used to communicate a change in volume, Justin uses "how much" to describe an instantaneous *rate* of change of volume, and "over time" to describe gradual change in water level over a period of time. Just as in clip 260, "how much" was coded as *rate of change in volume*. "Water level change over time," became the flagship member of the newly formed code: *change in volume over time.*

Clip 666

In clip 666, Justin reads the directions for part (b) of the Quabbin Reservoir task, including his own interpretation of the directions.

666 Justin: [reading] "Explain the changes in the **quantity** of **water**," or in other words, our graph that we just sketched, our millions of **gallons** graph, our "**volume**" graph [making quotation marks in the air with his fingers], "in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year. How are these changes evident in your graph in part (a)?"

Clip 666 includes four of my hypothesized *volume* vocabulary words: "gallons," "quantity," "volume," and "water." In this clip, Justin is reading the instructions to the task. The wording of the task is "the changes in the quantity of water," but Justin rephrases the task statement, cueing his interpretation or substitute language with "in other words." He offers three sets of "other words," which are "our graph that we just sketched," "our millions of gallons graph," and "our volume graph." "Gallons" was identified as a substitute, in Justin's case, for "volume," and vice versa. Both were originally coded as *volume,* as was "quantity of water." However, the combination of the concept of *volume* and the modifiers "changes in," or "graph," resulted in my recoding these phrases, and this clip, as *change in volume over time*.

Clips 253 and 254

Clips 253 and 254 below are an example of Justin's use of the term "displacement." He says that the graph of "quantity of water" would be "like a displacement graph." Because Justin does not refer to a given displacement at a given time, but in the sense of displacement changing ("going up and down") with respect to the horizontal time axis of the graph, the use of "displacement" as an analog for volume in clip 254 was coded as *change in volume over time* rather than simply *volume*.

253	Justin:	Alright, well, [part] (a) says "sketch a possible graph of the **quantity of water** in the reservoir as a function of time."
254	Justin:	So that *[the graph asked for in part a]* would be the dis-, that would be like a **displacement** graph, right? **Quantity of water**, whether it's going up and down [raising and lowering hand with the palm facing downward].

Clips 253 and 254 also further demonstrate Justin's use of the term "quantity." When reading aloud the instructions of the task, Justin uses the term "quantity." His only other uses of the term follow immediately after reading the term from the task, and in every one of these instances, he offers substitutes or synonyms for quantity. In clip 254, he uses "displacement graph" to express his interpretation of "graph of the quantity of water." Justin's quantity-evasion habit can also be observed in his Justin's offering of "other words" in clip 666 (shown previously).

The codes: *volume*, *rate of change in volume*, and *change in volume over time*, (shown in Table 2) developed as a result of this comparative search process for terms that I anticipated might be related to the concept of "volume."

Table 2
Three Emerging Concept Codes

Code	Description	Examples
Volume	The amount of volume in the reservoir. Often at a given point in time.	"the amount of water in the reservoir," "the quantity of water," "millions of gallons," "volume level," "the volume of the water"
Rate of change in volume	An instantaneous rate of change.	"amount of water that is coming in," "how much the water level is changing"
Change in volume over time	Describes a change in volume over a period of time, but does not attempt to quantify a rate.	"volume graph," "changes in quantity of water," "displacement (graph)"

I was surprised to find that "amount," and "how much" were used by the participants to refer to a rate of change in volume in interpreting the information given in the inflow and outflow graphs, rather than the amount of water in the reservoir at a given time. I had previously anticipated that the frequently used terms of "rate of flow," "derivative," and "velocity," would be more likely candidates to fill the "rate of change" role. My next step was to investigate the role of those terms. I searched the transcript for clips that included two or more of the following terms: change, flow, gallons per day, inflow, outflow, rate, rate of change, rate of flow, and velocity. As before, I analyzed the clips containing two or more of these terms to compare and contrast their use. I found that some of these terms were used to describe two *separate rates of change* while others described a net rate of change. When terms were used to describe rate of change as a net rate of change they were coded, as before, as *rate of change in volume.* I also found that, similar to the code *change in volume over time*, the participants also described a changing rate of change without attempting to quantify the rate at which the rate was changing. These instances were coded as *change in rate of change over time*. At times the participants also spoke of an instantaneous *rate of change of rate of change* in terms of "acceleration."

The lengthier of the transcript segments described earlier (Segments 3, 4, 9, and 14) all involved interval-by-interval descriptions of one or more of three specific graphs. First, there was the original graph of inflow and outflow that was given in the task. The participants used the graph given in the task to create two new graphs. One was a created graph of the net rate of change for the reservoir. The other created graph was a volume graph. I coded the participants' references to each of these graphs as either *original graph, rate of change graph,* or *volume graph*, respectively. Specific points on these graphs also became the subject of much discussion. I coded references to these points as *rate of change = 0, rate of rate of change = 0,* and *extrema*. For navigational purposes (to help me know when the participants were pointing out or referring to specific parts of the graph), I also coded references to specific *points and parts* on the graphs as well as *signs* which were adjectives that appeared in contrasting pairs that seemed to play important roles in the negotiation of meaning.

Refining the Concept Codes

A table with descriptions and examples of the concept codes can be found in Appendix E. After creating this table, I then collaborated with other members of the research team to verify my concept codes. We were not only able to refine these codes, but also were able to define relationships between many of the concept codes and vocabulary codes. The three concept codes of *separate rates of change*, *rate of change in volume*, and *volume*, and the corresponding references to the *original graph*, *rate of change graph*, and *volume graph* came to represent three frames of references for categorizing vocabulary that the participants used to interpret the different intervals of the Quabbin Reservoir Graph. For example, the concept of *net rate of change=0* from the frame of *rate of change* was concurrently interpreted as either *no change in volume* in the frame of *volume* or *inflow=outflow* in the frame of *separate rates of change.* These corresponding interpretations, along with related codes (in italicized text) and student language (in quotation marks) are shown in Table 3. These frames of reference eventually became a powerful explanatory factor for another set of emergent codes, the Language Awareness Codes.

Table 3
Organization of Concept Codes and Vocabulary to Reflect Frames of Reference

Frames of Reference		
Separate rates of change *Original graph*	*Rate of change* *Rate of change graph*	*Volume* *Volume graph*
Inflow = outflow "Where they (inflow and outflow) meet"	*Net rate of change = 0* "Zero points"	*No change in volume/Extrema* "Top and bottom points on volume," "leveling off"
"The inflow is greater than the outflow"	"Positive velocity"	*Change in volume over time* "Increasing in volume"
"The inflow is less than the outflow"	"Negative velocity"	*Change in volume over time* "Decreasing in volume"

Language Awareness Codes

As mentioned earlier, I also coded clips in which there appeared to be strong evidence of the participants' own awareness of the choices of language and representation that they were making in discourse. I originally used one code, "language awareness," but this code gradually developed into six other codes. One of these codes is Justin's use of "airquotes" (described as "making quotation marks in the air with his fingers" in clips 281, 339, and 666 shown above). I couldn't explain why Justin was making these "airquotes" at the time, or even see a pattern in their use. However, I considered them an important element of describing the negotiation of meaning in my data because they seemed to represent an underlying commentary by Justin in respect to his choice of, and others' preferences for, specific words and phrases. Justin and the other participants' hesitations in speech, requests for definitions and clarifications, and verbal cues such as "in other words" and "so that would be" led to the eventual development of the Language Awareness Codes shown in Table 4.

Example of Building Narratives through Codes

It was initially difficult to explain the presence of these language awareness codes, and I knew that explaining the choices made in discourse would involve a complex synthesis of differing variables present in the context of the discourse. For the purpose of analysis, I narrowed my focus to the occurrences of language awareness codes in the language of Justin in Segment 9: The Gospel According to Justin. Each occurrence of a language awareness code pointed me to the surrounding discourse via the relevant concept codes and vocabulary codes. This analysis then allowed me to draw conclusions about Justin's choices in discourse as essentially choices between social, mathematical, and experiential factors.

Segment 9: The Gospel According to Justin, begins as Justin and Daniel ask Julie "how she is doing" and Julie admits that she is not sure "where [they're] going." In the following portion of transcript, various pronoun codes can be identified. Justin and

Table 4
Descriptions and Examples of Language Awareness Codes

Language Awareness Code	Description	Examples (Justin is the speaker for each example.)
"airquotes"	Justin's practice of making quotation marks in the air with his fingers.	386: And so you know your slope of your [making quotation marks in air with fingers] "volume graph," the slope is going to be negative, right?
value judgment	Using language to comment on language as good, bad, appropriate, or inappropriate	413: . . . it's a really **bad way** to say it but that's the only thing I can think of.
hesitation	Hesitation in speech that includes pauses or repetition but no change in vocabulary.	351: So, this is, this is what I came up with [Justin's net rate graph in his notebook].
substitutes and synonyms	Providing elaboration, definitions, examples or alternative language to communicate meaning for precise vocabulary. Usually involves <u>parallel structure</u> or <u>verbal cues</u>.	264: <u>This is</u> **inflow**, so <u>this is</u> **how much water is coming in** . . . 666: "Explain the **changes in the quantity of water**," <u>or in other words</u>, **our graph that we just sketched** . . .
labeling	Providing a precise name for a concept. Generally follows the form **X** <u>would be</u> **Y**. May be considered the "inverse" of *substitutes and synonyms* because *labeling* generally involves a progression from more descriptive to more precise language, while *substitutes and synonyms* generally describes or interprets precise language.	327: . . . **how fast you're changing your volume** <u>would be</u> **the rate of flow**, right? 649: -and so <u>**it**'d</u> [*the anti-derivative*] <u>be</u> **your distance graph** [touching his volume graph]. So **this** [*anti-derivative*] <u>would be considered</u> **our volume graph** [Justin's volume graph].
word search	A middle category between *hesitation* and *substitutes and synonyms*, *word search* involves language that communicates dissatisfaction with the original language, and offers different language, as though the speaker is correcting himself.	355: And so the water, **volume of** the water is gonna stay the same 254: So that [*the graph asked for in part a*] would be the dis-, that would be **like a** displacement graph, right?

Daniel exhibit a literal use of "you" (330, 331) in referring to Julie's personal progress on the task. Julie uses "I" and "we" (332) in the literal sense as she admits that she does not understand where the collective group is metaphorically "going." Justin shifts between "we" and "I" (333) to indicate that he is giving his personal perspective on the direction of group activity. Justin also uses "you" in the general sense as he explains a process for combining the inflow graph and the outflow graph (338, 339).

From this point in the text on, portions of transcript are presented in five columns. The columns show, from left to right, (1) the clip number, (2) the video time code, (3) the speaker, (4) spoken text (with my interpretive annotations in [*italic text*]) and (5) the speaker's concurrent physical gestures and pointing to graphs and inscriptions. The fifth column of transcript is eliminated in portions of transcript in which physical gestures and pointing were not noted.

330	(0:31:22.4)	Daniel:	Julie where you at?	
331	(0:31:23.5)	Justin:	How you doing Julie?	
332	(0:31:25.5)	Julie:	Um, I don't I don't know. I, I still don't understand where we're going.	
333	(0:31:32.9)	Justin:	What we're gonna do, let's see, is, this is the way I see it, alright? This is the gospel according to Justin.	
334	(0:31:40.7)	Jamie:	[laughs]	
335	(0:31:41.4)	Justin:	Kay, so we're given this, this graph right here right?	[Justin indicates the original graph on Julie's page. It is right side up for Julie, but upside down from his point of view]
			It gives us an outflow graph	[tracing outflow graph roughly from left to right with pencil tip]

			and an inflow graph.	[tracing inflow graph from right to left with pencil tip]
336	(0:31:48.0)	Julie:	Right.	
337	(0:31:48.5)	Justin:	Now, to me, you can't really do much when you want to know how, what the volume of the water is, with those two graphs separate.	
338	(0:31:54.2)	Justin:	So, what I'm thinking to do is to add them [*inflow and outflow*] together, so you take the difference between the two points, right?	
339	(0:32:03.3)	Justin:	So like, so you take the, you start, start with the income, uh, inflow I'm sorry, the inflow and you subtract the outflow from that part right, that's gonna give you the amount of water that's either "coming in" or "leaving," if it's negative it's leaving if it's positive it's, it's coming in.	[tracing the vertical axis on the original graph between the inflow and the horizontal axis] [airquotes]
340	(0:32:23.7)	Justin:	Does that make sense?	

Of particular interest in this first portion of "the gospel" is Justin's use of what I have labeled "airquotes." I think of these "airquotes" as a way of qualifying one's choice of words. An example in written text would be *my* use of quotation marks around the phrase "the gospel" in this paragraph. In the strictest sense of the term "gospel," I would not consider Justin's explanation of his work on the Quabbin Reservoir Task a "gospel." I have observed that the term "gospel," generally applies to religious texts, specifically, the first four books of the New Testament. Justin's explanation is not one of the first four books in the New Testament. However, Justin's explanation does share some qualities of the four Gospels in the New Testament. For example, Justin's explanation is a first person narrative of an event. Also, Justin's narrative, like the first four books of the Old

Testament, is not meant to be a synopsis or summary. Rather, it is a detailed account. In other words, it is long. I quote Justin when I call his explanation “the gospel,” but my use of quotation marks has a purpose beyond avoiding plagiarism. I also use quotation marks to imply that I do not necessarily agree that “the gospel” is the most correct term for what I am describing, but that I believe it is a term that efficiently communicates an idea that might otherwise require an ambiguous label such as “Justin’s Explanation,” or a lengthy label such as “Justin’s explanation of his work on the Quabbin Reservoir Task.” Simply referring to this portion of transcript as “the gospel,” is more efficient (although later on I began referring to this segment simply as Segment 9).

The reader may have noticed that I also placed quotation marks around the term “airquotes.” Similar to “the gospel,” I use these quotation marks to qualify my choice of vocabulary. I originally coded Justin’s uses of “airquotes” as “making quotation marks in the air with his fingers.” I found this label to also be lengthy, and not necessarily an accurate description of what I was trying to describe. While working with other members of the research team, we began to refer to Justin’s gesture as “airquotes.” Therefore, I use quotation marks to qualify language that has been effective in my past attempts to communicate an idea, yet is language that doesn’t necessarily seem academically appropriate.

I have just given two examples of how my use of quotations marks in this written text is a way of implicitly admitting that I do not believe that I am necessarily using the most precise language, but am pragmatically using language that has proven to be effective in my past efforts to communicate. The data suggest that Justin’s use of airquotes in “the gospel” is also a sign of his conscious choices of effective language that he perhaps recognizes as not completely appropriate. Returning to the transcript, Justin demonstrates his use of airquotes in clip 339:

339	(0:32:03.3)	Justin:	So like, so you take the, you start, start with the income, uh, inflow I'm sorry, the inflow and you subtract the outflow from that part right, that's gonna give you the amount of water that's either "coming in" or "leaving," if it's negative it's leaving if it's positive it's, it's coming in.	[tracing the vertical axis on the original graph between the inflow and the horizontal axis] [airquotes]

It was not initially apparent why Justin would need to qualify his use of "coming in," or "leaving." These are common terms that appear appropriate for the situation. These terms had also been used previously in the mathematical discourse with little resistance or question. Confused as to why Justin had now began to qualify his use of "coming in" and "leaving," I viewed this moment of confusion as an opportunity for me to learn something new about my participants' reasons for their choices (Leatham, 2006). Guided by an assumption that there was a reason for Justin's use of airquotes I embarked on a process of data-guided analysis to develop an explanation for Justin's choice.

Having used my language awareness code of "airquotes" to identify "coming in" and "leaving" as vocabulary of interest, I next used my vocabulary codes to investigate portions of transcript where Justin and the other participants had used similar vocabulary. I then compared the concept codes associated with the portions of transcript and found that the participants had used similar vocabulary to communicate slightly different concepts. Different language awareness codes and events helped me to understand subtle shifts in language that corresponded to these differences in use. These subtle shifts in language became the basis of narratives of the negotiation of meaning for a term that Daniel had originally referred to as “zero points.”

I followed a similar process of investigating Justin’s use of language awareness codes in Segment 9: The Gospel Accoring to Justin by tracing the interaction of the relevant vocabulary and concept codes to unravel the negotiation of meaning in mathematical discourse. The result is three connected narratives of the negotiation of meaning and language for (1) points of inflection and concavity, (2) “zero points” (critical points), and (3) the analogical problem solving process that related the language of “velocity” and “displacement” to “rate of flow” and “volume.” These three narratives are presented in the next chapter.

CHAPTER 6: RESULTS AND FINDINGS

My analysis of "The Gospel According to Justin," revealed, in Justin's language, echoes of previous segments of discourse. In tracing these echoes back to their sources, I uncovered in my data the negotiation of meaning for three conceptually important ideas that may be characterized in the conventional language as (1) hypercritical points, (2) critical points, and (3) the analogical mappings between contextualized examples of a derivative. In the language of our participants, the negotiation of meaning involved the terms (1) "inflection" and "concavity," (2) "zero points," and (3) "velocity" and "volume." The negotiation of meaning for these three conceptually important ideas was interwoven, all finding root in Daniel's language in Segment 3, and all being echoed in Justin's language in Segment 9. Between Segments 3 and 9, Jamie, Julie, and the instructor, Dr. Walter, drive the negotiation of meaning with questions and comments about language. In this section, I provide three narratives for the negotiation of meaning for these three conceptually important ideas, describing my findings in terms of the three agentive explanatory factors of personal experience, mathematical understanding, and social roles; as well as social and egocentric speech; and conventional and ordinary language.

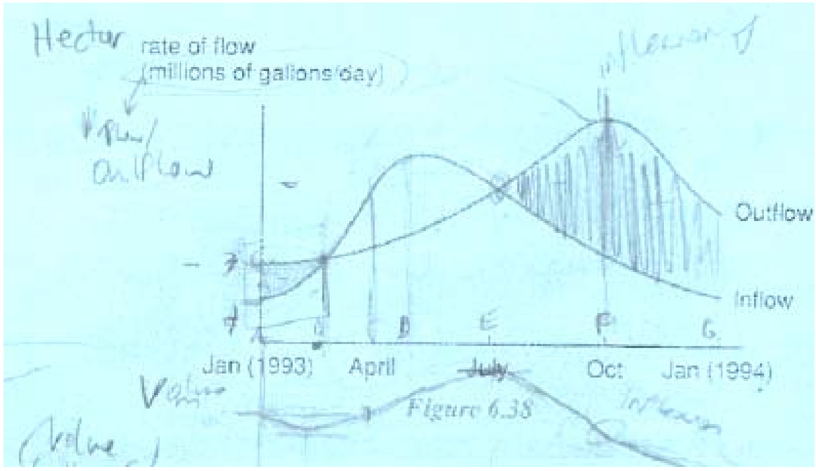
Inflection Points: Personal Experience and Personal Pronouns

Two different definitions for the conventional term *inflection points* are negotiated, one of inflection point as *extrema* of the *rate of change* and another of inflection point as the point between concave down and concave up intervals. Daniel introduces the idea of inflection points as extrema (Segment 3), and, as a response to questions from Jamie (Segment 4) and Julie (Segment 5), uses pronouns and the ordinary language of personal experience to demonstrate how extrema in the original graphs of inflow and outflow and his created rate of change graph are reflected in the shape of the volume graph. Julie asks how the inflection points are related to “the concave,” another term that she has heard in previous discourse, and Justin builds on Daniel’s definition, using Daniel’s words and inscriptions, to connect meaning for “inflection point” to meaning for “concave.” We gain insight into Justin’s visual meaning for “concave” from a question he asks the instructor, and then observe Justin use the terms “concave” and “inflection” in “the gospel” (Segment 9)

Daniel Identifies an “Inflection Point” (Segment 3)

When Daniel introduces the term “inflection point,” he speaks of inflection points as *extrema*, saying that an inflection point is where the slope is the greatest. Daniel identifies an inflection point on the original graph, and then on the volume graph. While Justin’s choice was to first construct a net rate of change graph from the original graph, and then construct a volume graph, Daniel’s gestures reflect how he originally constructed his volume graph directly from the original graphs of rates of inflow and outflow. Later, upon noticing how the other participants created net rate graphs as an intermediate graph between the original graph and the volume graph, Daniel also constructed a net rate (or “velocity”) graph of his own (Figure 8).

58b	(0:13:50.1)	Daniel:	And it has, its greatest slope is right here,	[point F, October, on the original graph]
			so that’s its inflection point	[point F, October, on the volume graph].



Original Graph

Volume Graph

Rate Graph

Figure 8. Daniel's graphs.

A "Normal" Inflection Point (Segment 4)

When Daniel describes the shape of the volume graph to Jamie, his description of an inflection point changes. He formerly described an inflection point as the point of "greatest slope," but now he identifies an inflection point as "when it has the most outflow," terminology that reflects the idea of *separate rates of change*. Looking to the volume graph as a representation of water level, Daniel also describes an inflection point as where the volume graph "comes down and starts to level off."

106	(0:17:05.3)	Daniel:	It goes negative	[tracing the rate graph from E to F]
			and then that point up there	[point F on the original graph]
			is when it has the most outflow, I believe.	
107		Jamie:	Um hmm.	

108	(0:17:14.9)	Daniel:	That's your inflection parts. That's when it goes like whoop.	[Daniel draws a line with negative slope which first becomes more negative and then more positive]
			Hmm. Like that part right there. So that's, like, then it comes down	[the volume graph from E to F]
			and starts to level off, kind of, like the water level.	[the volume graph from F to G]

Although Daniel has just described what a point of inflection looks like on a volume graph, Jamie asks Daniel why she can't see the point of inflection on his volume graph.

109	(0:17:29.1)	Jamie:	Okay so why isn't this inflection point	[point F on the original graph]
			really reflected anywhere on this part of the graph?	[circling the area around point F on the volume graph]

A point of highest velocity, or highest rate, is a salient feature as the maximum on a rate graph. To the untrained eye, however, a point of inflection is not a particularly salient feature on the graph of a function. Daniel recognizes that inflection points are "hard to draw," and emphasizes the shape of an inflection point by drawing a prototypical inflection point (Figure 9). Although Daniel has been speaking primarily in the third person "it" to refer to the graph, he now personifies the graph, using the first person "I." Daniel, as if speaking for the graph, says, "I'm going down down down" and then "I'm leveling off." Daniel's falling intonation dramatically recreates the sensation of falling down.

110	(0:17:34.6)	Daniel:	It, this is hard to draw, like, a normal inflection would be like, "I'm going down, down, down."[falling intonation]	[drawing a line with a negative slope that becomes more negative (Figure 9)]

111		Jamie:	[laughs]	
112	(0:17:40.3)	Daniel:	And then it goes, “I’m leveling off.”	[continuing the line with the negative slope becoming less negative]

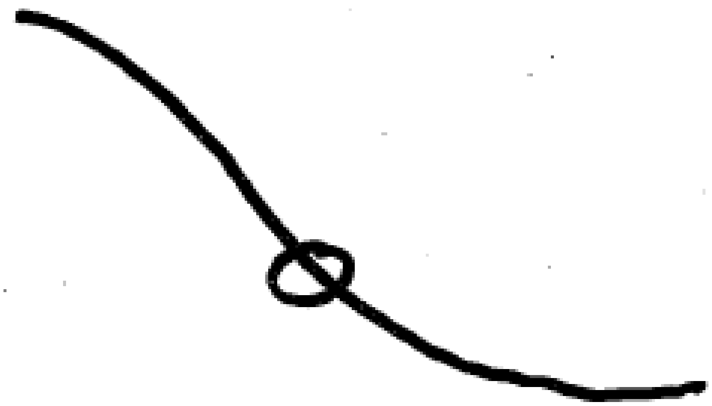


Figure 9. Reproduction of Daniel’s drawing of a “normal” inflection point.

Daniel summarizes his example by circling the point of inflection (Figure 9), describing the point as “where it just goes down and then starts to level off,” and then describing of a point of inflection in the language of *extrema* as “the highest velocity that you’ll have.”

113	(0:17:43.3)	Daniel:	So it’d be like that. Like, that would be the inflection point.	[drawing a circle (Figure 9) where the slope stops decreasing and begins increasing]
114		Jamie:	Kay.	
115	(0:17:46.3)	Daniel:	Like the point where it just goes down and then starts to level off.	[drawing another line and circling the inflection point]
116	(0:17:50.8)	Daniel:	This is the highest velocity that you’ll have.	

It is not until Jamie attempts to identify the same inflection point on the *rate graph* that Daniel admits that an inflection point can also be where the velocity is the lowest, demonstrating a more complete view of inflection points of a function as *extrema* of the derivative of the function.

124	(0:18:17.8)	Jamie:	So is this inflection point	[F on the original graph]
			[inaudible]?	[pointing to the corresponding portion of Daniel's volume graph]
125	(0:18:20.0)	Daniel:	Oh yeah. so I guess that it should be the most negative here, be like that, and then it starts, like that's the lowest velocity it ever gets.	[drawing a minimum at October on the rate graph]

The Slide Example (Segment 5)

Julie, who generally works patiently on her own, but often asks important clarifying questions about the group's approach to the task, asks Daniel to clarify what he means by point of inflection. Daniel hesitates at first as to how he should structure his explanation, but then refers to the common human experience of riding on a playground slide to help Julie see what he means by an inflection point being "where the velocity is the highest."

141	(0:19:13.7)	Julie:	I don't think I understand the inflection idea.	
142	(0:19:19.1)	Daniel:	Okay, so if you're, like, here's kind of an, idea, okay. So if you're drawing, a curve. Um, like, you just. Ah.	
143	(0:19:32.9)	Daniel:	Okay, so the inflection point is where the velocity is the highest,	
144	(0:19:37.3)	Daniel:	so, like, if it, if you were like going on a slide and if you're falling down on it, your speed would be	[Drawing a curve similar to Figure 9, shown in Figure 10]

			increasing, you'd be going like down really really quick-	
145		Julie:	Um hm.	
146	(0:19:48.3)	Daniel:	Then at some point you'd start to level off. And where would your velocity be highest?	
147	(0:19:53.4)	Julie:	Right there. Before you [inaudible]	[pointing with pencil]
148	(0:19:55.7)	Daniel:	Yeah, like, before you start to slope off.	[pointing with his pencil, Figure 10]
149	(0:19:59.2)	Daniel:	Yeah, that's just the inflection point. Like where the velocity is highest.	
150		Julie:	Okay.	

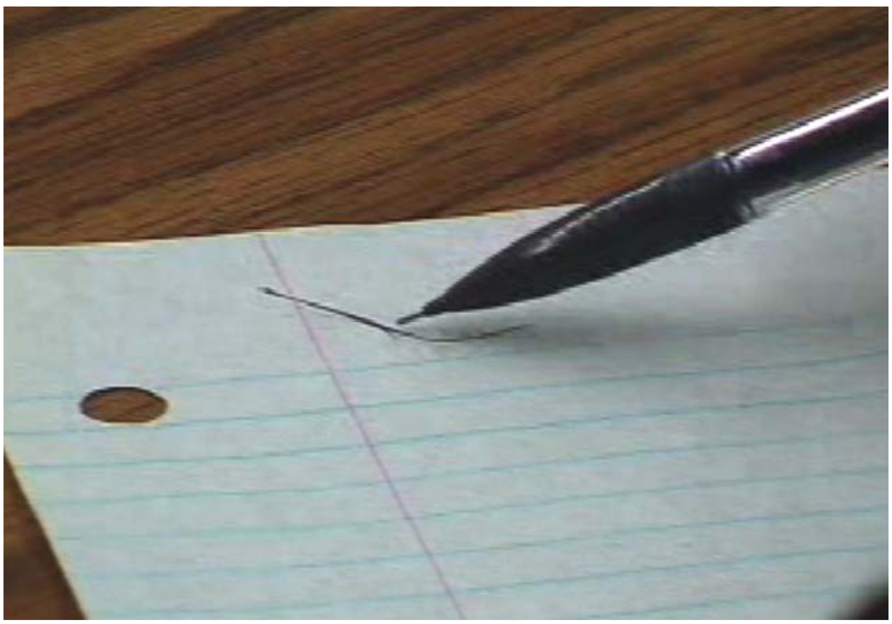


Figure 10. Identifying "the point where you start to level off."

In a process of linguistic invention (Walter & Johnson, 2007), Daniel skillfully combines his meaning for the mathematical concept of a point of inflection as extrema of the derivative or "velocity" graph with a common human experience in which "highest velocity" can be pinpointed in the form of a kinesthetic sensation. There is an element of iconic interpretation (Leinhardt, Zaslavsky & Stein, 1990), as the shape of the inflection point that Daniel draws also looks like the shape of a playground slide. I do not consider this a negative form of iconic interpretation, as it is not so much the shape of the curve, but Daniel's definition of an inflection point as where "the velocity is the highest" that justifies the location of the inflection point.

Daniel's use of the second person pronoun "you" (144-146) may be interpreted as a general form of "you" that is replaceable with the general "one" as in "if one were going on a slide." However, the sense of the pronoun is not entirely general, because Daniel also holds Julie as the responsible referent of "you" when he asks her, "where would your velocity be the highest?" Because of the personal nature of the embodied sensation of maximum, I view this use of "you" as a more personal use, intended to invite Julie to participate in recalling her own personal experience. In doing so, Daniel demonstrates his awareness of Julie as a participant in such an experience.

Drawing a Curve (Segment 5)

Daniel offers another example to help Julie recognize how the point of inflection as an extreme value of rate or slope (velocity) is reflected in the shape of a function. He invites her to draw a curve and "feel the point where you start to curve off," offering Julie an additional opportunity to "feel" a point of inflection within the convenience of the mathematics classroom. Again, Daniel's use of "you" may be in a general sense, as if anyone can draw the curve and feel the inflection point, but in the context of an invitation to act, his use of "you" takes on a slightly more personal sense. Once again, Daniel animates his action, speaking in the first person as if he were the curve, or riding down the curve as a slide.

151	(0:20:05.0)	Daniel:	Like if you were to draw, like a line, like you can kind of feel the point where you start to curve off.	[drawing more decreasing curves similar to Figures 9 and 10.]
152		Julie:	Right.	
153	(0:20:13.0)	Daniel:	Cause you're like drawing and AHHH!	
154		Julie:	[laughs]	
155	(0:20:14.4)	Daniel:	Like "I'm falling" and then "whew!"	

Connecting Inflection to Concavity (Segment 5)

Julie uses Daniel's description of inflection point to attempt to connect it to "concave," another term that has emerged in conjunction with the term "point of inflection" in a recent whole-class discussion. Julie asks, "And so, when you were talking before, the inflection point did, did the concave, right? Did it start the concave or was that the point of . . ." Julie's speech fades off, and Daniel admits that he does not remember "the concave thing." Jamie chimes in with a rhyme about concavity, "concave down like a frown, concave up like a cup." Laughing at Jamie's rhyme, possibly because he may have learned a similar pneumonic device in his own previous calculus class, Justin helps Daniel to understand the idea of concavity in terms of points of inflection.

165	(0:20:41.0)	Justin:	[laughing at Jamie's rhyme] So concave down, like that,	[drawing a parabolic curve opening down on the curve Daniel has drawn (Figure 11)]
			that's concave down.	
166		Daniel:	Um hm.	
167	(0:20:44.3)	Justin:	-and this is concave up,	[drawing a parabolic curve opening up on the curve that Daniel has drawn]
			and so they [*concave down and concave up*]-	
168		Daniel:	Ohh!	

169	(0:20:46.3)	Justin:	-change at that point of inflection.
170		Daniel:	Ohhh! That's smart.
171	(0:20:48.6)	Justin:	So that's what you're try-, that's what you're saying, the point, that's, the veloc-, the velocity's the highest at that point of inflection on a displacement graph.
172		Daniel:	Oh, okay.
173	(0:20:58.0)	Julie:	So it's where they change.

Figure 11. Justin's addition of concave up and concave down "parabolas" to Daniel's point of inflection.

The most remarkable portion of this section of transcript is the way that Justin builds on Daniel's previously stated ideas on inflection. Rather than create a new diagram to demonstrate concavity, Justin highlights the concave down and concave up portions that already exist in the representation created by Daniel (Figure 11). Furthermore, although Justin adds the idea that concave down and concave up "change at that point of inflection," he also returns Daniel's words to him in his explanation, saying "that's what you're saying . . . the velocity's the highest at that point of inflection on a displacement graph." Justin's use of "you" is literal, because Daniel literally has been saying that a point of inflection is where "the velocity's the highest." Although Justin does not always identify whose language he is repeating in further discussion as he does here, he does continue to echo his peer's language in the negotiation of meaning many times in this study as evidence of his awareness of, and meaning for, his peers.

Justin's Question: Concave Right? (Segment 5)

Justin provides additional insight into his understanding of concavity when he turns to his instructor and asks about the correct way to characterize the concavity of a parabola opening to the right.

182	(0:21:22.6)	Justin:	Just out of curiosity Dr. Walter, if they had a, if you have a curve like this, would it be concave right?	[shows Dr. Walter a curve he has drawn that looks like a parabola opening to the right]
			Or is it still concave down	[the top half of the curve]
			and concave up?	[the bottom half of the curve]

The instructor explains that, although the curve that Justin has drawn does not represent a function, the concavity of the curve would most likely be described as concave down and concave up. Justin thanks the instructor and says no more on the subject. His question, however, is interesting in that the idea of "concave right" portrays concavity as a physical characteristic of a curve, with no explicit connection to slope or rate of change. Had the participants attempted to interpret concavity in terms of velocity or rate of change, the notion of "concave right" for a parabola opening to the right may have led to an interesting discussion. As this discussion did not happen, I can only say that Justin's meaning for concavity was related to the visual shape of a parabolic curve opening in a certain direction.

Clip 182 demonstrates another unique use of pronouns in Justin's language. He speaks of "they" when he says, if "they had a curve," but corrects his language, replacing "they" with "you." Analysis of others instances of Justin's use of "they" and "you" reveals that Justin generally uses the animate "they" to refer to the creators of the Quabbin Reservoir Task, as in "they've given us this graph." Justin most often uses "you" in the general sense in explaining mathematical activity as in "you subtract this distance." This instance of "you" may be an exception to his general use, as Justin specifically replaces "they" with "you." A possible interpretation may be that Justin identifies Dr. Walter with the creators of the task and so the referent of "they" becomes "you" as Justin realizes that the person to whom he was addressing his question may be the literal referent of "they." Another possible interpretation is that Justin uses "they" to refer to the community of mathematicians, and recognizes Dr. Walter as a member of that community. Although there is not sufficient data to draw a firm conclusion as to why Justin substitutes "you" for "they," this may be an example of an instructor playing the role of the more experienced participant in mathematical discourse who is better able to inform Justin as to how conventional language of mathematics is used (Lampert, 1990).

Inflection and Concavity in "the Gospel" (Segment 9)

Later, as Justin is using the net rate graph to explain the shape of the volume graph to Julie in Segment 9, he interweaves the two definitions for points of inflection, referring to Daniel's idea of *extrema* in *rate of change* (velocity) when looking at the rate graph, and building on his own idea of switching back and forth between concave up and concave down intervals on the *volume graph* (displacement graph) (Figure 12).

399	(0:36:39.4)	Justin:	So it's [*the volume graph*] going concave up right here	[after the first minimum on the volume graph (Figure 12)]
			because it [*the rate graph*] keeps on getting higher and higher and higher	[after first x-intercept on net rate graph]
			and so it [*the volume graph*] keeps on raising faster and faster and faster	[approaching the first point of inflection on the volume graph]
			until it [*the volume graph*] gets to that inflection point, this point right here.	[first inflection point on volume graph, first maximum on rate graph]
400	(0:36:49.6)	Julie:	Okay.	
401	(0:36:50.6)	Justin:	And then all of the sudden, it's [*the volume graph*] still rising,	[right after the first maximum on the rate graph]
			but it's going, it's rising gradually slower and so it [*the volume graph*] starts concaving down, right?	[right after the first inflection point on the volume graph]
402	(0:36:58.6)	Julie:	Okay.	

Just as Daniel used the repetition of "down, down, down" to emphasize the idea of increasing speed in his earlier description of an inflection point in the context of a playground slide, Justin repeats, "higher and higher and higher" to portray increasing values on the rate graph, and correspondingly, "faster and faster and faster" to convey the idea of increasing rate of change, or slope, on the volume graph. Justin describes this interval of "concaving up" on the *volume graph* as leading up to the inflection point, which he identifies as *extrema* on the *rate graph*. Following the inflection point, decreasing rate, observed as a decreasing interval on the *rate graph,* is described as "concaving down" on the *volume graph*. More examples of Justin's descriptions of the shape of the volume graph[1] can be observed in the full transcript of Segment 9, found in Appendix F.

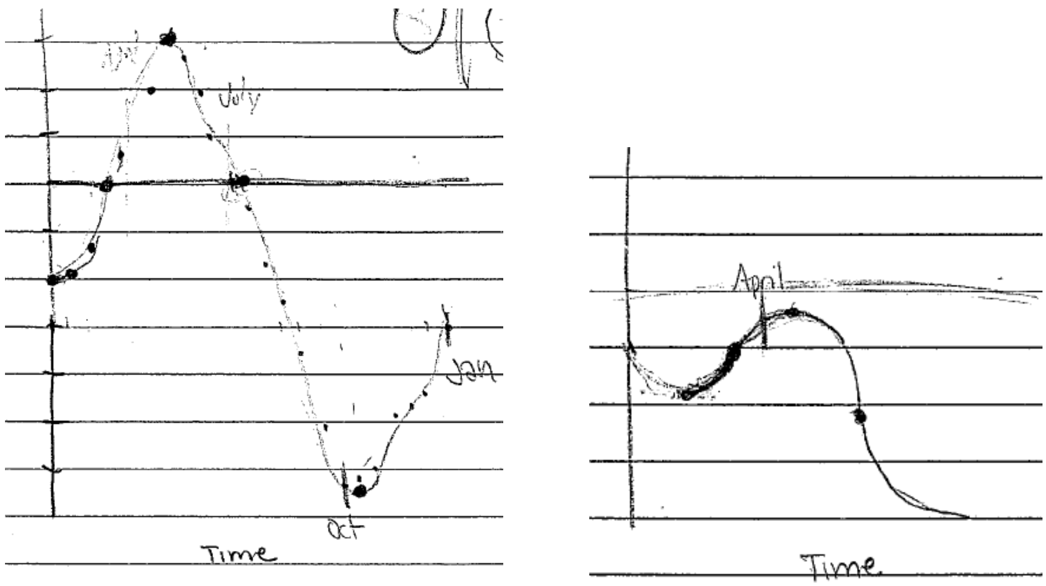


Figure 12. Justin's "velocity" (rate) graph (left) and "displacement" (volume) graph (right).

Zero Points: Shifting Frames of Reference to Reflect Experience

In Chapter 5, I described how my initial investigation of Segment 9 drew my attention to Justin's use of airquotes in the phrase "the amount of water that's either 'coming in' or 'leaving'" (339). I followed my vocabulary codes, tracing Justin's use of "coming in" to Daniel's use of similar vocabulary in the negotiation of meaning for the

[1] Justin's explanation of the second inflection point is not as smoothly navigated, and at the end of his explanation, Justin comments that his language "is a really bad way to say it." This final portion of the graph may be one of the more difficult portions to describe because the volume is decreasing while the rate is increasing. If the volume were increasing and the rate increasing, one could say that the volume is increasing at an increasing rate. Here, however, it is difficult to determine whether one should say that the volume is decreasing at an increasing rate or decreasing at a decreasing rate. In the sense of becoming more positive, the rate is increasing. In the sense of directionless rate, or speed, one might say that the rate is decreasing because it is getting closer and closer to a rate of zero. (For example, one may be driving their car in reverse slower and slower and slower). Similar student reactions to the concept of signed velocity are reported in Johnson (2005) and Nemirovsky (1994).

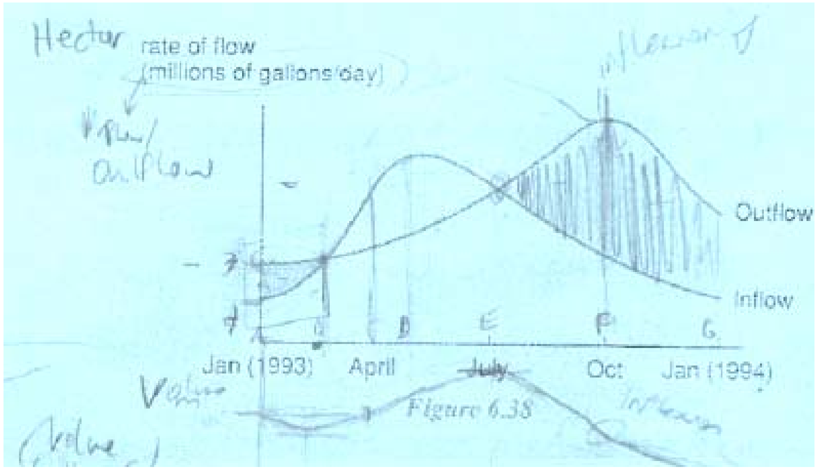
term “zero points” (Segment 6). This comparison of vocabulary and concept codes revealed that the frames of reference of *separate rates of change, rate of change in volume,* and *volume* played a role in how this vocabulary was interpreted by the participants. Here, I tell the story of negotiation of meaning in chronological order, beginning with Daniel’s initial recognition of the importance of “zero” (Segments 3 and 4) and offering the sequence of data that led me to an eventual interpretation of Justin’s use of airquotes in “the gospel.”

Daniel Identifies a Zero (Segments 3 and 4)

Returning again to Segment 3, we first see Daniel speak of zero in his statement that the volume graph “levels off at zero.”

58a	(0:13:50.1)	Daniel:	So, it has a negative slope. And then it starts going positive up to that point.	[point E, July, on the rate graph]
			And so it levels off at zero.	[point E, July, on the volume graph]
			Cause the v-, the v- [1 sec]	
			I don’t know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. [2 sec]	[point E on the volume graph]
			Which is correct on our velocity chart.	[point E on the rate graph]

Although he has difficulty explaining exactly *what* is zero, Daniel recognizes that the zeros of the rate graph play an important role in the shape of the volume graph. The idea of leveling off at zero reoccurs often in Daniel’s language, accompanied by various conventional terms that might be associated with the idea of rate of change (Zandieh, 2000), such as “tangent” and “slope,” and some less conventional language such as “rate of positivity.”



Original Graph

Volume Graph

Rate Graph

Figure 13. Daniel’s graphs.

91	(0:16:09.4)	Daniel:	And then it hits zero, so like your tangent there would be zero.	[volume graph at B]
			And then it starts going to positive to this point.	[volume graph from B to E]
			Like it stops going positive here.	[rate graph at E, July]
			Like this is the rate of positivity . . .	[rate graph from B to E]

97	(0:16:37.7)	Daniel:	-then it starts to go to a zero in positivity in its growth.	[rate graph from D to E]
98		Jamie:	Okay.	
99	(0:16:43.0)	Daniel:	So it, it slopes out.	[volume graph at E]

Coining and Clarifying “Zero Points” (Segment 6)

Daniel coins the term “zero points,” as he helps Julie identify the intercepts of her rate graph. Daniel is much more detailed, focusing the negotiation of meaning, as he helps Julie to recognize what a zero point looks like. Using the original graph as a guide in creating the volume graph, Daniel identifies the points where the inflow and outflow meet as zero points. The equality of inflow and outflow results in the water level remaining constant. “So,” reasons Daniel, “where the inflow and the outflow meet are . . . zero points.”

201	(0:22:49.3)	Daniel:	. . . Okay, this [point E on the original graph] is where the inflow equals the outflow, right?
202		Julie:	Um hm.
203	(0:23:00.0)	Daniel:	And so the velocity there equals zero, right? Cause like, there’s neither inflow, there’s not water coming in, nor is there water coming out.
204	(0:23:15.6)	Julie:	Right.
205	(0:23:15.7)	Daniel:	I mean, like there is, but they’re equal, so the water level isn’t actually changing.
206	(0:23:20.5)	Julie:	Okay. So how’s this gonna look?
207	(0:23:25.0)	Daniel:	So like where the, where the outflow and the inflow meet, are gonna be our zero points.

Although Daniel is being much more explicit about the correlation of the original graph and the volume graph, he still has not explicitly identified what the zero points have to do with “zero.” At one point (203) he states that the “velocity” is zero, but then corrects himself, saying that zero velocity does not necessarily imply zero outflow and zero inflow (203, 205). Dr. Walter, who, with the exception of answering Justin’s question about concavity (182), has been a silent observer of the conversation, enters the conversation by revoicing Daniel’s statement and asking Daniel to clarify what is zero at a “zero point.”

208	(0:23:34.3)	Dr. Walter:	I don’t understand what you mean, where the inflow and the outflow meet you’re gonna have zero points. Zero points of what?

In overlapping speech, Daniel and Justin offer different language for describing zero points. Daniel appears to offer “velocity” as a synonym for “the level of water overall,” but then stops to think (209). Justin (210) revoices Daniel’s previous language

(203, 205) of “where the inflow and outflow meet” to say “where the inflow and outflow are the same,” and draws the same conclusion as Daniel, that such a relationship in *separate rates of change* will be reflected in the frame of *change in volume over time* as “no change in water level.” Justin and Daniel verbally co-construct the idea (215-216) that the “velocity,” or net rate graph, demonstrates “zero” because the graph crosses the horizontal “x-axis.” This co-constructed idea is justified in two different ways. Justin takes a local approach, saying that “the water level won’t change at that point” (217). Daniel makes a more holistic statement about the sign of the velocity (rate) graph and the resultant increasing and decreasing shapes of the water level (volume) graph (216-220). Using a form of case elimination, Daniel explains that if positive rate means that the water level is increasing, and negative rate means that the water level is decreasing, then no change in water level must fall in between negative and positive rate, giving a zero rate.

209	(0:23:42.7)	Daniel:	The level of water overall. So the velocity. I think, let’s see.
210	(0:23:52.1)	Justin:	Well it’d [*a zero point*] be where the inflow and the outflow are the same, so there is no change in water level.
211		Daniel:	Yeah.
212	(0:23:57.2)	Justin:	So,
213	(0:23:59.0)	Daniel:	Like where they meet, though.
214		Justin:	you don’t change your veloc-
215	(0:24:00.3)	Justin:	-yeah where they [*inflow and outflow*] meet, yeah, cause they’d be the same. So in a velocity graph it’d be where they would, it would cross the x-axis.

216		Daniel:	x-axis, yeah, cause like the, uh	
217	(0:24:07.3)	Justin:	Cause the water level won't change at that point.	
218	(0:24:11.9)	Daniel:	-velocity of the water, if the velocity is positive,	
			it will be increasing in water level.	[sweeping hand to the right and upward]
219	(0:24:18.5)	Daniel:	And if the velocity is negative,	
			it will be decreasing in water level.	[sweeping hand downward and to the right]
220	(0:24:22.5)	Daniel:	And so, when the, when there is no change in the water level for a certain time,	[holding both hands at the same level]
			the velocity will be zero of water coming in or water coming out.	[moving right hand away and then toward himself]

To describe what is zero about a zero point (220), Daniel uses the language of water "coming in" and water "coming out." This language is problematic for Jamie because it seems to imply that both inflow and outflow, or *separate rates of change*, are also zero at a zero point. Jamie tells Daniel that she doesn't agree that the separate rates of change have to be zero at a zero point.

221	(0:24:36.0)	Jamie:	[3 sec] I don't agree with that. The velocity could be equal.	
222	(0:24:42.9)	Daniel:	Kay, like how so?	
223	(0:24:46.2)	Jamie:	The velocity of the water coming in equals the velocity of the water going out.	[moving right hand toward herself] [moving left hand away from herself]
			So your [3 sec] your, your water level in your reservoir is going to stay the same.	[raising and lowering hands with palms down]
224		Daniel:	Yeah.	

225	(0:25:01.0)	Jamie:	Whereas, if you had a higher velocity coming in than going out, your water level is going to rise.	[raising hand with palm down]
			If you have a lower velocity going in than going out, your water level is going to drop.	[lowering hand with palm down]
226	(0:25:12.7)	Daniel:	So if the velocity of coming in and going out were the same, what would the total velocity equal?	
227	(0:25:18.8)	Jamie:	They would be equal.	
228	(0:25:20.8)	Daniel:	Wouldn't it be zero? Like if you have a negative velocity, of like, five, that's coming out	[sweeping right hand away from self and to the right]
			and a positive velocity of five that's coming in -	[bringing left hand toward right hand in front of self]
229	(0:25:30.5)	Jamie:	Going in. That's true.	
230	(0:25:32.7)	Daniel:	-wouldn't that be zero?	
231	(0:25:33.9)	Daniel:	But we could also like draw these two different functions as separate, like we've been drawing velocity together-	

Daniel and Jamie seem to agree that a zero point (a net rate of zero) does not necessarily imply that no water is entering or exiting the reservoir. (In fact, in clips 203 and 205, Daniel recognized the same fact that Jamie is working to help him understand here.) The negotiation of meaning taking place stems from a difference in frames of reference revealed by language. Daniel's statement in line 220 that "the velocity will be zero of water coming in or water coming out," immediately follows his reference to zero on the velocity (or net rate graph) which would be in the frame of a net *rate of change in volume,* later referred to as "total velocity" (226). Jamie, however, interpreted Daniel's statement as a claim about the *separate rates of change*, and found the claim of zero velocity to be false. This difference in frames of reference is further evidence by Jamie's use of the plural pronoun "they" for *separate rates of change* (227) and Daniel's use of the singular pronoun "it" for a net *rate of change in volume* (228). At the end of this exchange, Daniel suggests that their difference in opinion might be resolved by drawing the two functions separately, in accordance with Jamie's idea of *separate rates of change* rather than "drawing velocity together" as a net *rate of change*.

Language for Separate Rates of Change (Segment 7)

The negotiation between Jamie and Daniel resulted from Daniel's description of net rate of change with the words "water coming in" and "water coming out." Before creating an explanation for Justin's later use of airquotes with the words "coming in" and "leaving" in Segment 9, I first examine Justin's use of similar vocabulary. As Justin reiterates the goals of the task in Segment 7, he uses the term "how much water is coming in" to describe the inflow graph. He and Jamie then construct a direct interpretation between the frames of *separate rates of change* and *change in volume over time*, stating that "if the inflow is smaller than the outflow . . . the reservoir is going down."

264	(0:26:52.5)	Justin:	Yeah, well this is, this is, yeah,	
			this	[inflow on the original graph]
			is the inflow right here. This is inflow, so this is how much water is coming in-	
265	(0:26:58.1)	Jamie:	Uh-huh.	
266	(0:26:58.3)	Justin:	-right here.	[tracing the inflow line on the original graph]
267	(0:26:59.9)	Justin:	So	
268		Jamie:	So-	

269		Justin:	-if the inflow is greater than, is smaller than the outflow, that means that the reservoir is-	
270	(0:27:04.5)	Jamie:	going down.	
271	(0:27:05.4)	Justin:	-going down.	[bringing right hand down to meet left hand]

Explaining the Airquotes (Segment 9)

In the previous transcript, (264-271), Justin shifted from the *separate rates of change* language ("how much water is coming in," 264) to language from the frame of *change in volume over time* ("the reservoir is going down," 269, 271), in order to explain the combined effects of inflow and outflow. Later, in the opening lines of "the gospel," we see Justin's first recorded attempt (339) at describing the combined effects of inflow and outflow in terms of *rate of change* language. He uses airquotes to qualify his language choice.

339	(0:32:03.3)	Justin:	So like, so you take the, you start, start with the income, uh, inflow I'm sorry, the inflow and you subtract the outflow from that part right, that's gonna give you the amount of water that's either "coming in" or "leaving," if it's negative it's leaving if it's positive it's, it's coming in.	[tracing the vertical axis on the original graph between the inflow and the horizontal axis] [airquotes]

Based on the discussion between Jamie and Daniel in Segment 6, a possible reason for qualifying the use of "amount of water coming in" for a positive net rate may be the fact that such language does not tell the entire story. A positive net rate may be, and in the case of the Quabbin Reservoir, *is*, a way of summarizing that there is more inflow than outflow. However, speaking of a positive net rate as merely "how much water's coming in" may imply that there is only inflow, or that outflow is zero. Speaking of negative net rate as “how much water is leaving” also only tells half the story, and would be more appropriately interpreted as “how much more water is leaving than is coming in.” The mathematical practice of taking the difference in inflow and outflow and interpreting that difference as a net rate results in the loss of information about outflow and inflow.

Furthermore, Justin’s failure to find ordinary language that would express a net rate of change more appropriately in terms of the reservoir suggests that “net rate of change” is not something that is often encountered in his experiential history with bodies of water. This is an example of Justin’s language reflecting a choice between correctly communicating his understanding of the mathematics and correctly representing his personal experience. The mathematical reality of combining two functions (inflow and outflow with respect to time) and interpreting the result as a net rate of change is at odds with experience, because the inflow and outflow of a reservoir happen at two different locations in the reservoir. Inflow is a measure of the rate of water entering the reservoir, which may be observed where the water source for the reservoir meets the reservoir. At another location on the reservoir, a mechanism of sorts is constructed to moderate the amount of water that is permitted to leave the reservoir, either for a designated use or to avoid an overflow. It would be impractical, however, to build a reservoir so that water enters and exits the reservoir at the same location. Therefore, while one can observe the amount of water “coming in” to a reservoir (inflow) and the amount of water “leaving” a reservoir (outflow), one does not observe the combined effects of these two rates as a net rate in any physical location.

However, one can observe the combined effects of inflow and outflow in terms of the *change in volume over time* as the level of water rises and falls. Justin's language reflects this phenomenon as he moves to the frame of *change in volume over time* in his next attempt to interpret the combined effects of inflow and outflow.

342	(0:32:25.4)	Justin:	So, if you did that [*subtraction of outflow from inflow*] just over, you know, just did that for every single part, this would be that part that's leaving,	[shading the area between inflow and outflow from A to B (Figure 14)]
			this is the, uh, water coming in, this is the, when the, um, water,	[the area between inflow and outflow from B to E]
			"volume" level is rising.	[airquotes]
			This is when it's [*the volume level*] going down again.	[the area between inflow and outflow from E to G]

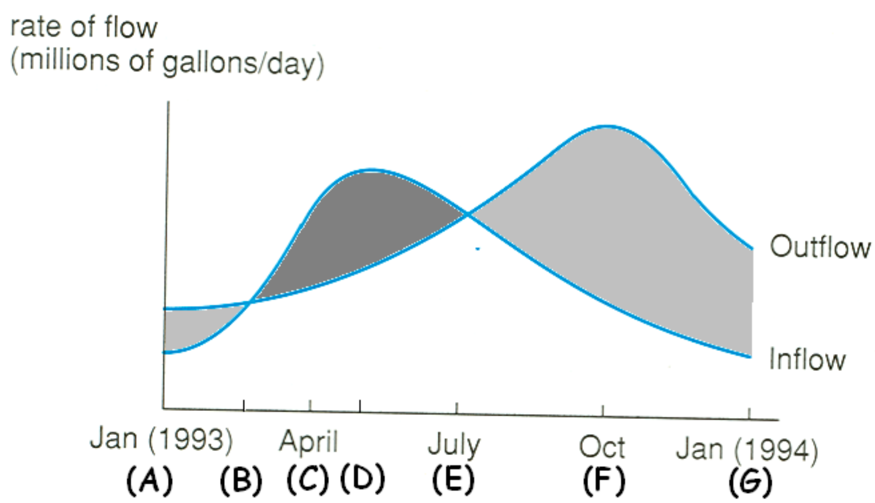


Figure 14. Justin shades the area between the inflow and outflow graphs.

In clip 342, Justin appears to have found a compromise between his experience of viewing inflow and outflow in separate locations and the mathematically accepted practice of combining inflow and outflow to find a net rate of flow by shifting his frame of reference to one where a net rate, interpreted as net change over time, can be more easily and accurately mapped to human experience.[2]

In summary, Justin's choice to refer to a positive net rate of change as "the amount of water that's either 'coming in' or 'leaving,'" is his response to a situation where the ordinary language that has been used to describe the mathematical idea of a net rate of change does not fit well with common human experience.[3] He uses airquotes to qualify his choice of language, and eventually, through a shift in frames of reference (from *rate of change in volume* to *change in volume over time*), is able to find language that fits both his experience and the mathematics. In clip 349, we see Justin again interpret a signed net rate in terms of *change in volume over time*, demonstrating that he has not abandoned the frame of *net rate of change*, but has merely abandoned the practice of interpreting a net rate in terms of water "coming in" or "leaving."

349	(0:33:09.9)	Justin:	And so it's [*the result of the subtraction*] gonna give you a negative flow rate. Or in other words, the water, the, the "volume" of the water is lowering, right?	[airquotes]

[2] While Justin appears more comfortable with his new interpretation, it should be noted that he has not lost his practice of using airquotes. He now qualifies his use of the word "volume" as a substitute for "water [level]." I address Justin's continued use of airquotes with "volume" in the third narrative, "Velocity vs. Volume," which begins on page 112.

[3] In a related note, the participants demonstrated a tendency to speak of zero points as points of "no change in water level" for a given *period* of time even though the term "zero point" implies that no time is passing. I view this as yet another example of the explanatory factor of experience clashing with the explanatory factor of mathematical understanding. Experientially, very few entities undergo instantaneous change. It is even difficult to think of something "staying the same" for one point in time, as the word "stay" implies the passage of time. However, the mathematical limiting process allows the creation of abstract constructs such as "instantaneous rates of change" which imply that something can "be changing" (or in this case, not changing) at a point. Rather than interpret zero points as "points of no change," which experientially may be a viable description of *every* point in time, the participants often referred to zero points as "periods of no change" as in lines 220 and 223. The limiting process of a derivative involves exchanging a series of periods for a point. While the linguistic exchange of a point for a period may be counter-productive from this limiting perspective, it seems quite acceptable in the present discourse for its loyalty to human experience.

In a final example, we see Justin coordinate the three frames of reference as he describes a zero point. Rather than immediately connecting the frame of *separate rates of change* to *rate of change in volume*, in a way that would represent his original solution process (Figure 15), Justin's language again follows the course of *separate rates of change* interpretation ("the same amount of water is coming in as it is leaving"), which leads into a *change in volume over time* interpretation ("the volume of the water is gonna stay the same"). Finally, Justin uses his *change in volume over time* interpretation to make a claim about *net rate of change* ("the rate of change will be zero").

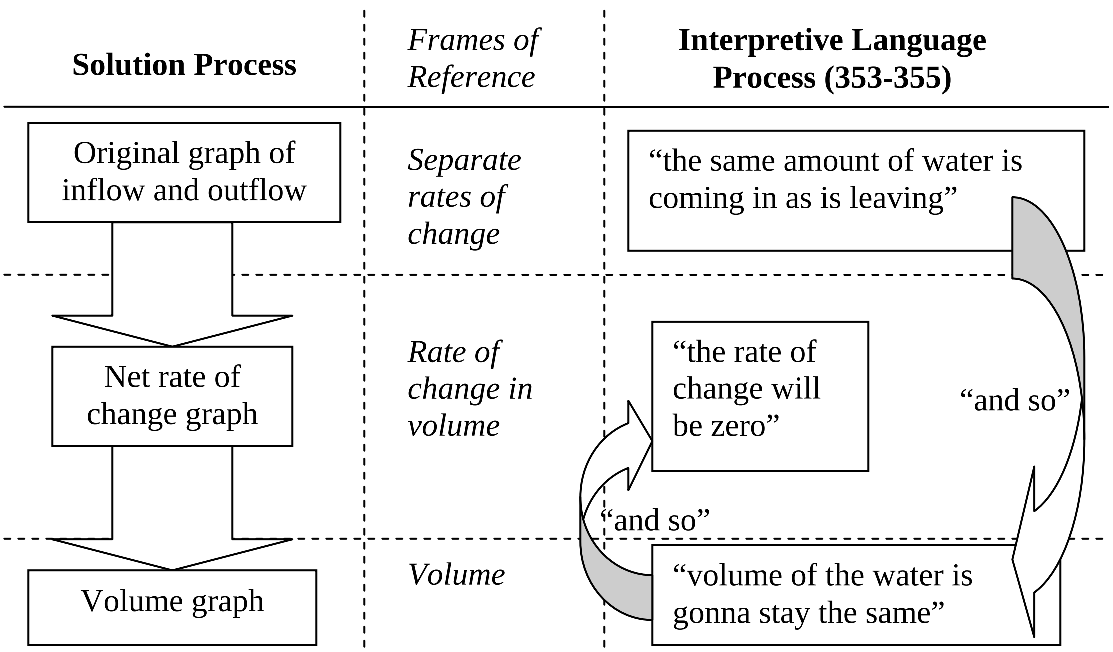


Figure 15. Comparing Justin's language with his solution process.

353	(0:33:37.7)	Justin:	So at that point,	[first x-intercept on rate graph]
			this point right here, and at this point right here,	[the two points where the inflow and outflow intersect on the original graph]

			the same amount of water is coming in as it is leaving, right?	[moving both hands across the table at the same rate]
354	(0:33:45.9)	Julie:	Right.	
355	(0:33:46.5)	Justin:	And so the water, volume of the water is gonna stay the same.	[holding arms out wide with palms in as if running them along the surface of a giant sphere]
			And so, it's um, the rate of change will be zero, does that make sense?	

Justin appears to have found a reasoning sequence that not only fits his understanding of the mathematical relationships between *separate rates of change*, *net rate of change*, and *change in volume over time* at the zero points, but also one that fits into his experiential frame, allowing him to offer a semantic warrant (Walter & Johnson, 2007) of a physical interpretation for his claims about the mathematical relationships.

Velocity vs. Volume: Analogical Reasoning Revealed

I have previously noted how Daniel, Jamie, and Justin speak of rate of change as "velocity," and Justin uses airquotes with the term "volume." My analysis suggests that these two phenomena are related in that they are evidence of the participants' analogical reasoning about the Quabbin Reservoir Task. In the terminology of analogical problem solving, the participants applied the ideas of their previous work in which they have used velocity graphs to create displacement graphs (base problem) in order to complete the Quabbin Reservoir Task (target problem). Having solved the problem of relating the graph of a function to the graph of the function's derivative in the velocity-displacement context, a next step would be mapping the velocity-displacement context to the Quabbin Reservoir Context. This would include mapping a velocity function to a net rate (or "rate of flow") function and a displacement function to a quantity function. If the two problems are isomorphic, the next step in analogical problem solving could be to apply the solution of using slope to relate a function to its derivative in the new context, thereby completing the problem.

However, the Quabbin Reservoir Task was not completely isomorphic to previous velocity-displacement tasks, and two modifications were required in order to apply the solution of the base problem to the target problem. First, the direction of the solution is reversed as the Quabbin Reservoir provides information about the derivative function and requires students to reconstruct the anti-derivative. Second, the information about the derivative, or rate of change, is not given directly. Rather than supply a net rate of change graph, the designers of the task supplied graphs of inflow and outflow, requiring the participants to produce a graph of net rate before isomorphic mapping, and an eventual solution, could be reached. A possible analogical solution process for the participants' work on the Quabbin Reservoir Task is shown in Figure 16. Vertical lines represent mathematical relationships that were used in the solution process. Horizontal lines represent the mapping of one context to another. For the participants in this study, this mapping was never entirely explicit, but was revealed in the participants' ongoing search for labels for their various graphs.

A large portion of the participant's negotiation of meaning and language can be explained in terms of this model of analogical problem solving. Near the beginning of the transcript (Segment 2), Justin quickly summarizes the solution process for the Quabbin Reservoir Task.

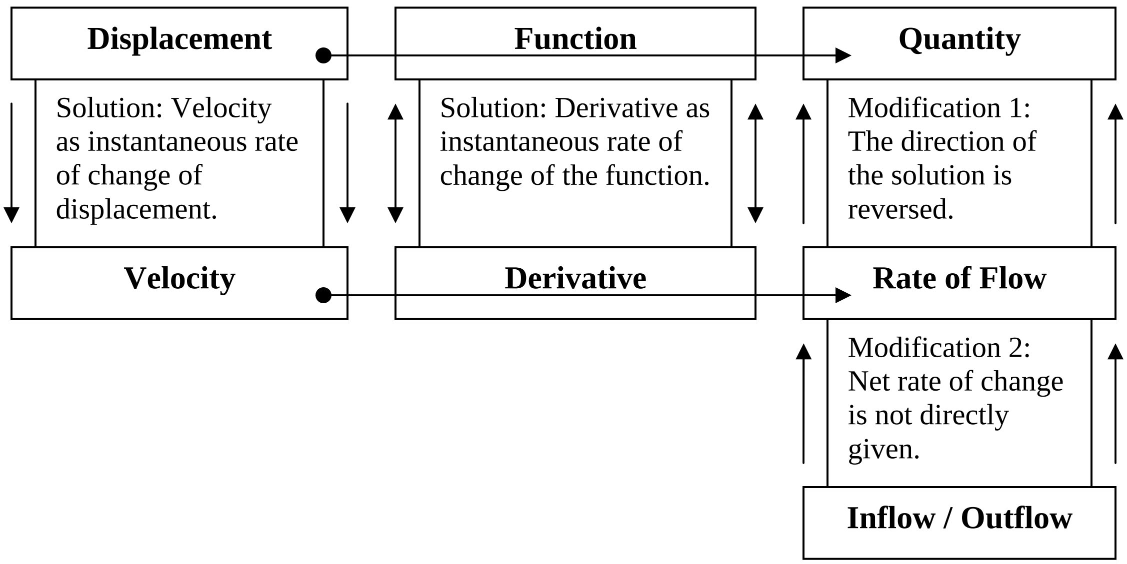


Figure 16. An analogical solution mapping for the Quabbin Reservoir Task

42	(0:12:48.0)	Justin:	Okay. Kay so cause I, I was thinking about what you did is that you took, you worked backwards.	[turns to Daniel]
43		Daniel:	Yeah.	
44	(0:12:55.5)	Justin:	Cause this is the rate of flow. So this is what the derivative will look like when you add those [*inflow and outflow*] together. Kind of like what I did.	[the original graph] [the original graph]
45		Daniel:	Yeah, that's what-	
46	(0:13:02.0)	Justin:	But then you guys worked backwards and decided what the, what the flow, what the water level would look like, the distance.	[raising and lowering pencil held horizontally in the air]
47		Daniel:	Yeah.	
48		Justin:	So.	
49	(0:13:11.0)	Daniel:	Does everyone understand that? Because I didn't explain very well.	
50		Julie:	You just added, er-	

Justin combines the conventional term “derivative” with the ordinary language “work backwards” to describe the first modification required to complete the Quabbin Reservoir Task. Justin also refers to the second modification, saying that they can see what the derivative looks like by combining the inflow and outflow graphs. Although Daniel seems to understand Justin’s abstraction of the analogical solution process, there still remains much to be negotiated. In fact, the bulk of the content of Segment 9: The Gospel According to Justin, is an explanation of the two modifications shown in Figure 16. Having already described in detail how he combined the inflow and outflow graphs to create a net rate graph, Justin explains his idea of “working backwards.”

389	(0:35:45.4)	Justin:	There’s gonna be some point, ‘cause, we’re working backwards.	
			Instead of finding the derivative,	[pointing to volume graph and sliding pencil up to the net rate graph]
			we’re going from the derivative	[net rate graph]
			backwards.	[sliding pencil to volume graph]
			Trying to, trying to figure out how to go backwards, right?	[pointing first at the net rate graph and then sliding the point of the pencil to point at the volume graph again]
390	(0:35:54.9)	Justin:	And so if we have that point, zero on the derivative,	[first x-intercept on the net rate graph]
			that means that on the original graph, that point is level, the tangent line is zero. There is no slope. Make sense?	[first minimum on the volume graph]
391	(0:36:07.3)	Julie:	Yeah.	

What is interesting about this portion of transcript is that it is the only time that Justin uses the term “derivative” in Segment 9: The Gospel According to Justin. This introduction of the conventional term “derivative,” juxtaposed with the ordinary language of “working backwards,” may function to allow Justin to introduce a new procedure as mathematically appropriate. The participants have worked on numerous tasks and exercises in which they “found the derivative,” and, in that sense, have negotiated procedural and conceptual meaning for the term derivative. Here, Justin refers to the familiar process of “finding the derivative” to create meaning for the new process of “going from the derivative backwards.”

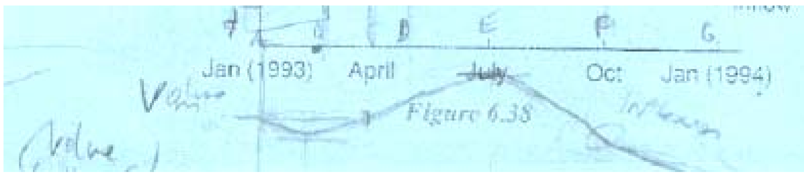
The details of relating a function to its derivative have been partially explained in terms of critical points (zero points) and hypercritical points (inflection). While inflection points were described primarily in the language of “highest velocity,” from the base context of velocity and displacement, reasoning about zero points mixed the language of “velocity” and “inflow and outflow.” In this section, I trace the progression of language from “velocity” to “volume” as the participants negotiate the mapping of language in the analogical solution process.

“The Velocity of the Flow of the Water or Something” (Segment 3)

Once again, I return to the clip from Segment 3 in which Daniel expresses doubt about his use of the term velocity. He not only hesitates, but explicitly acknowledges that he is not sure what to call the analog of velocity in the Quabbin Reservoir context.

58a	(0:13:50.1)	Daniel:	So, it has a negative slope. And then it starts going positive up to that point.	[point E, July, on the rate graph]
			And so it levels off at zero.	[point E, July, on the volume graph]
			Cause the v-, the v- [1 sec]	

I don't know what you call that. The velocity of the flow of the water or something? The velocity of this is zero. [2 sec]	[point E, July, on the volume graph]
Which is correct on our velocity chart.	[point E, July, on the rate graph]



Volume Graph

Rate Graph

Figure 17. Daniel's graphs.

It is apparent from Daniel's gestures that when he says, "velocity chart," Daniel is referring to his net rate graph. Although Daniel qualifies his use of "velocity" here, he, Jamie, and Justin soon begin speaking metaphorically, using "velocity" to refer to net rate of flow and "displacement" to refer to quantity or volume.

Why Velocity? (Segment 7)

As Justin, Jamie, and Daniel offer interpretations for zero points (Segment 6), they continue to speak of their rate graph as their "velocity" graph. During a brief pause, Julie, who has not spoken the word "velocity" in respect to the Quabbin Reservoir Task thus far, remarks, "I think I'm still confused with the idea that it's velocity." Jamie admits that she is also confused about the velocity idea. Justin and Daniel suggest that they should call what they have been calling velocity "rate of flow." Julie's response reveals that her confusion about velocity will not be cleared up by a simple change in language, "But we're not trying to find the rate of flow, aren't we trying to find quantity?"

234	(0:25:44.9)	Julie:	[3 sec] I think I'm still confused with the idea that it's velocity.
235	(0:25:50.8)	Jamie:	Yeah, me too.
236	(0:25:51.9)	Justin:	Kay just call it [*what they have been calling velocity*] rate of flow then.
237	(0:25:53.3)	Daniel:	Yeah. Rate of flow.
238	(0:25:54.0)	Justin:	We should call it [*velocity*] rate of flow.
239	(0:25:55.1)	Julie:	But we're not trying to-
240	(0:25:55.3)	Daniel:	Rate of flow's easier.
241	(0:25:56.1)	Julie:	-find the rate of flow-
242	(0:25:56.2)	Justin:	I know we're tryin-
243	(0:25:56.9)	Julie:	-aren't we just, trying to find quantity?
244	(0:25:58.2)	Jamie:	Yeah.
245	(0:25:58.9)	Daniel:	Yeah.
246	(0:25:59.1)	Justin:	Umhm.
247	(0:26:00.1)	Daniel:	But like here-
248	(0:26:02.2)	Justin:	But this graph right here gives us the rate of flow, and so we-
249	(0:26:04.4)	Jamie:	[to Julie, laughing] That was a good face!
250	(0:26:05.4)	Julie:	[sighs]
251	(0:26:07.7)	Daniel:	Like [pause]
252	(0:26:09.6)	Jamie:	Hmm. I think we should find what we're trying to, trying to get at here.

Justin's response to Jamie's suggestion (252) is to re-read the instructions as stated in the task. He then offers his interpretation of the given instructions, referring to the quantity of water as being "like a displacement graph." He makes his analogical language more explicit, correcting his metaphorical statement that a graph of quantity would be displacement by adding the word "like," thus converting his metaphor to a simile (254).

253	(0:26:18.4)	Justin:	Alright, well, (a) says "sketch a possible graph of the quantity of water in the reservoir as a function of time."
254	(0:26:24.7)	Justin:	So that [*the graph asked for in part a*] would be the dis-, that would be like a displacement graph, right?

			Quantity of water, whether it's going up and down.	[raising and lowering hand with the palm facing downward]
255		Jamie:	Um huh.	
256	(0:26:31.1)	Justin:	Displacement.	[raises and lowers hand again]

It is uncertain whether Julie's concerns stem from a lack of recognition of the language mapping of "velocity" to refer to a rate of flow graph (the horizontal lines in Figure 16) or a deeper question concerning the rationale of the group's solution process thus far (the vertical lines in Figure 16).[4] Having expressed themselves that velocity is not exactly the most appropriate term for the Quabbin Reservoir Task, Justin and Daniel initially treat Julie's confusion as a language issue, and are more careful to show their consciousness of the possible confusion caused by the use of "velocity" and "displacement." Justin begins to use airquotes with the term "velocity," implying his recognition of the fact that the graph does not literally show "velocity" (the rate of change of displacement) and also that the use of the term is not necessarily accepted by all of the participants in discourse.

277	(0:27:11.1)	Justin:	So the, this would be our rate of flow or "velocity" if you will, right?	[the original graph] [airquotes]

Even though Justin has addressed the fact that "velocity" is not the most appropriate language Julie continues to press the issue that velocity, or rate of change, is "not what [they're] trying to find." Again, Justin rephrases the point of the task, which is to take the given information about the rate of change and describe the resultant changes in quantity, or water level, by creating a graph of quantity with respect to time.

[4] My follow up interview with Julie suggested that her confusion may have involved a combination of both vertical and horizontal factors.

280	(0:27:19.8)	Julie:	That's not what we're trying to find.	
281	(0:27:20.7)	Justin:	No, we're trying to find,	
			we're trying to take this graph,	[points to the original graph]
			they're telling us how much the water level is changing	[moving hands together and apart vertically with palms facing]
			and make uh, our "best guess"	[airquotes]
			at what the water level looks like,	[holding two hands with palms facing as before]
			a graph of how, the water level change over time. Does that make sense?	[drops the left hand and just moves the right hand slowly up and down with palm facing downward]

Units of Measure for Quantity (Segment 8)

Julie and Jamie appear to be more comfortable with the restated goals of the task. However, the normally silent observer Dr. Walter asks Justin to clarify the hand motions that he uses to accompany his statements about the change in water level over time (284).

284	(0:27:36.4)	Dr. Walter:	So you're thinking of measuring the quantity of water in the reservoir by the height [1 sec] of water in the reservoir?	
			When you're doing this I'm imagining you're talking about the height?	[raising and lowering flat hand]

Justin justifies his use of height as a matter of preference: height has fewer dimensions than volume to worry about (286).

286	(0:27:48.2)	Justin:	That's [*measuring volume by height*] how I, that's how I think about it, cause I don't know how else, I guess you could measure it [*quantity of water*] in, like volume, but, I don't know, height, just, to me, seems more, one, two dimensional.	[raising and lowering hand with palm down]

Although height may be more natural to work with, and a better fit for the graphical method of representing quantity, Daniel suggests that "volume" may be "better," or more appropriate for the situation. Dr. Walter suggests that the participants look at the units of measure associated with the original graph given with the task. Upon reading "millions of gallons per day," Justin concludes that volume is more appropriate than height for measuring the quantity of water in the reservoir.

292	(0:28:36.3)	Dr. Walter:	What's your, what's your rate measured in?
293	(0:28:38.0)	Justin:	Rate is measured in-
294	(0:28:39.1)	Jamie:	Millions of gallons per day.
295	(0:28:39.8)	Justin:	Millions of gallons per day, so it would be volume.

"What Would the Derivative of Volume Be?" (Segment 8)

Having been reminded that "velocity" may not be the most appropriate term for the present discussion, Daniel poses a question about the correct language for referring to the derivative of volume. He uses analogy to reason about language, noting that there is a special name for the derivative of displacement (velocity) and a special name for the derivative of velocity (acceleration). Daniel asks if there is a special name for the derivative of volume. Justin reasons that velocity is really just "how fast you're changing your displacement," and so "how fast you're changing your volume," should be "rate of flow."

318	(0:30:31.5)	Daniel:	[4 sec] So, cause like when you take the derivative of displacement, it's velocity.
319	(0:30:43.6)	Justin:	Velocity.
320	(0:30:46.4)	Daniel:	When you take that derivative it's um-
321	(0:30:47.8)	Justin:	Acceleration.
322	(0:30:48.2)	Daniel:	-acceleration, but what would the derivative of volume be?
323	(0:30:51.2)	Justin:	I would imagine it [*the derivative of volume*] would be rate of change, rate of flow, change.
324	(0:30:54.6)	Daniel:	Rate of change, yeah. [4 sec] Rate of flow change.
325	(0:31:03.3)	Justin:	Cause your velocity is just your rate of change of your displacement-
326	(0:31:05.1)	Daniel:	Yeah.
327	(0:31:06.1)	Justin:	-like how fast you're changing your displacement, so how fast you're changing your volume would be the rate of flow, right?

Although Daniel does not disagree with Justin's label for the derivative of volume, he also does not seem satisfied. For a major portion of Segment 9, Daniel searches the textbook, and eventually presents the idea that perhaps the derivative of volume can be represented as "surface area." This idea does not seem particularly applicable, and, concluding that his idea "doesn't really make much sense," Daniel says that perhaps the current language of "inflow" and "outflow" will suffice.

Shifting Gestures for Volume (Segment 9)

Despite Justin's statements that volume is a more appropriate term for describing the quantity of water in the reservoir, he nevertheless continues to use airquotes and other gestures with the term "volume" throughout "the gospel." This can be observed by revisiting clips from the previous discussion of zero points.

342	(0:32:25.4)	Justin:	So, if you did that [*subtraction of outflow from inflow*] just over, you know, just did that for every single part, this would be that part that's leaving,	[shading the area between inflow and outflow from A to B]

			this is the, uh, water coming in, this is the, when the, um, water,	[the area between inflow and outflow from B to E]
			"**volume**" level is rising.	[airquotes]
			This is when it's [*the volume level*] going down again.	[the area between inflow and outflow from E to G]

349	(0:33:09.9)	Justin:	And so it's [*the result of the subtraction*] gonna give you a negative flow rate. Or in other words, the water,	
			the, the "**volume**" of the water is lowering, right?	[airquotes]

A second look at these clips suggests that Justin uses "volume" as a replacement for "water." Subsequent uses suggest that "water" may be a cut-off version of "water level," as Justin leaves his airquotes to incorporate a palm down gesture similar to the one that he previously used to accompany the term "water level."

352	(0:33:20.9)	Justin:	they're [*inflow and outflow*] gradually becoming equal,	
			so it's the water level, the volume level is staying the same. Right?	[holding hand flat with palm down at eye level]

The last time that Justin used this palm down "water level" gesture, Dr. Walter questioned the gesture as demonstrating Justin's measuring of the amount of water in the reservoir by height (284), which led to Justin's eventual conclusion that "volume" would be a more appropriate means for measuring the amount of water (295). Justin's next use of volume (355) is accompanied by a gesture that is much more three-dimensional, possibly reflecting Justin's statement that height has fewer dimensions than volume, and his emerging recognition that the multi-dimensional "volume" may be more appropriate than height for measuring the quantity of water in the reservoir. Once again, he offers "volume" as a replacement for "water," which may or may not be a truncated version of "water level."

353	(0:33:37.7)	Justin:	. . . the same amount of water is coming in as it is leaving, right?	[moving both hands across the table at the same rate]
354	(0:33:45.9)	Julie:	Right.	
355	(0:33:46.5)	Justin:	And so the **water, volume of the water** is gonna stay the same.	[holding arms out wide with palms in as if running them along the surface of a giant sphere]

Justin summarizes how he combined the inflow and outflow graphs, and his purpose in doing so, in clip 366, stating that a graph of net rate will help one to understand the shape of the "displacement graph." However, he quickly inserts the substitute term, "volume graph," as though guilty of misspeaking, yet qualifies the term "volume" with airquotes, reflecting his consciousness, and possible qualification, of this alternative choice in language.

366	(0:34:22.7)	Justin:	. . . and so that kind of helps to combine the two graphs, like that, cause now you can see what the rate of flow, what the change of, in the flow rate, is, over time, and that kind of helps ya understand what's going on with the displacement graph.	[Justin traces his rate graph from right to left and then left to right three times as he speaks, finishing on the word "understand"]
			"Volume" graph	[airquotes]

Labeling the Volume Graph (Segment 9)

The Gospel According to Justin ends with a short discussion of units of measure for volume and a final identification of the important points that help to correlate the three graphs. Justin and Daniel have previously commented that the initial value of the volume graph is arbitrary. Although the rate graph gives enough information to sketch the correct shape of the volume graph, the actual values of the rate graph will differ by a constant from the true values for volume. Justin resurrects this previous conversation with Daniel as he explains this idea to Julie as "the thing you don't know."

429	(0:38:25.2)	Justin:	Now the thing you don't know, which is like kind of what Daniel said,	[pointing toward Daniel with his pencil]
			'cause I had that line right here.	[a horizontal line that Justin previously erased from his volume graph]
			You don't know where to start this graph at.	[touching points on the vertical axis of the volume graph]
430	(0:38:31.8)	Justin:	I mean, this graph, you know, this could be one thousand, um, cubic gallons, or whatever I don't know you say that.	[the y-intercept of the volume graph]
431	(0:38:38.2)	Justin:	Or it [*the initial value of the volume graph*] could be, you know, you don't know how, where to start this graph at.	[holding pencil parallel to the horizontal axis of the volume graph and sliding it back and forth in the "vertical" direction]

This transcript raises a question about Justin's meaning for the word "gallons." Here, and later, he modifies the term "gallons" to say "cubic gallons." The term "cubic gallons" may seem redundant to the listener, and Justin admits his doubts about his term by saying that he doesn't "know how to say that."

Although the other participants do not question the term "cubic gallons," Justin's hesitation associated with "cubic gallons" may imply an underlying process of Justin's negotiation of meaning with himself. Justin has modified his former one-dimensional gestures for volume as “water level” to three-dimensional gestures to accompany the term “volume level.” It may be that Justin’s use of “cubic gallons” is his way of verbally modifying a given unit of measure to create a unit of measure for volume. Just as “feet” become “cubic feet” and “meters” become “cubic meters” when speaking of volume, Justin applies this pattern to gallons to say “cubic gallons” for his volume graph, neglecting the fact that gallons by nature are a unit of volume. Justin continues to work to negotiate a relationship between the terms “volume” and “gallons.” In the next transcript (437-443), Justin re-labels his volume graph, which was formerly labeled “water level,” as “volume.” When Julie asks if volume really is the correct term for the graph of quantity, Justin refers to the fact that the given graph measures rate in gallons per day, and so keeping “the same kind of units,” the quantity graph would have to be a volume graph.

437	(0:38:52.9)	Justin:	. . . does that make sense?	
			how I went from, how I combined the two graphs	[pointing to the rate graph]
			and then how I went, how I looked at this graph	[rate graph]
			and then tried to make a, um volume.	[volume graph]
			Take away my “water level,”	[erasing a mark on the volume graph]
			that’s right this [*the label on the vertical axis of the volume graph*] should be “volume.”	
438	(0:39:08.1)	Julie:	[5 sec] Is it volume?	
439	(0:39:13.4)	Justin:	Yeah, ‘cause we’re working, it’ll [*the volume graph*] be the volume of the water.	[pointing to the volume graph]
440	(0:39:16.8)	Justin:	‘Cause this, the rate is in gallons, gallons per day.	[the rate graph]

441	(0:39:21.1)	Justin:	[3 sec] And so you just want to keep the same kind of units so it'd [*the units of the vertical axis of the volume graph*] be gallons, cubic gallons, I guess is what it would be.	[pointing to the volume graph]
442	(0:39:28.9)	Julie:	Oh. Oh, okay, yeah.	
443	(0:39:31.5)	Justin:	So volume.	

Julie makes one last summary of the critical points on the graph with Justin's help. Together, Justin and Julie demonstrate the dialogic phenomenon of exchanging discourse habits that at one time may have seemed unnatural to them (Lewis & Ketter, 2004). While Julie initially had doubts about the term "velocity," in clip 449 she uses it comfortably to communicate with Justin.

447	(0:39:38.4)	Julie:	No, I think that's, just like, the top and the bottom points are your zero points?	[pointing to points on her page]
448	(0:39:45.7)	Justin:	On, top and your bottom points for what graph?	
449	(0:39:49.6)	Julie:	Like these points on your volume graph	[minimum and maximum on her volume graph]
			are your zero points on your velocity?	[touching her rate graph]
450	(0:39:56.2)	Justin:	Yes.	[nodding]

Justin then points out the inflection points on both graphs. Like an instructor revoicing (Forman & Ansell, 2001) the words of his students, Justin replaces Julie's ambiguous ordinary language of "the top and the bottom points" with the more conventional terms "maximums" and "minimums." Although he applies his airquotes now to the term "velocity," he continues to use the equally questionable term "displacement" to talk about a volume graph.

452	(0:40:01.2)	Justin:	And then your maximums and your minimums on your veloc-,	[Julie's rate graph]
			your, we'll call "velocity" graph are going to be your points of inflection on your displacement graph.	[airquotes] [pointing to Julie's volume graph]

Re-Labeling (Segment 12)

Thus far, Justin has used gestures to place airquotes around "velocity" and "volume," has repeatedly replaced the terms "water" and "displacement" with "volume," and hesitated to speak of "cubic gallons." This hesitation and shifting language in for the concept of *volume* began when Dr. Walter asked Justin if he was thinking of measuring the volume of the water in the reservoir by the height (284). The process of verbal re-labeling has also been reflected in Justin's inscriptions, as Justin has replaced the original label of the vertical axis of his quantity graph, which read "water level," with the word "volume." Figure 18 shows Justin's attempts thus far to map "displacement" to an appropriate term in the Quabbin Reservoir Context. The dotted arrows in the figure indicate language that Justin has spoken but either abandoned or recognized as not completely appropriate. The solid arrows indicate language that Justin continues to use.

Justin's process of re-labeling continues as he again reads the instructions for part (a) of the Quabbin Reservoir Task.

622	(0:48:04.1)	Justin:	Kay, so, um, [part] (a) was, sketch our little graph, of the quantity of water in the reservoir as a function of time.	
			So it'd be volume,	[begins to erase the label on the vertical axis of his volume graph]

			so it'd [*the label of the vertical axis of the volume graph*] be gallons, cubic gallons, what is it, what is it, millions of gallons yeah, millions of gallons.	
623		Daniel:	It'd be millions of gallons, per day.	
624	(0:48:23.6)	Justin:	millions, well it'd [*the label of the vertical axis of the volume graph*] just be millions of gallons, yeah per-	[writing "millions of gallons" on the vertical axis of his volume graph]
625	(0:48:28.2)	Daniel:	Per day.	
626	(0:48:28.8)	Justin:	-per, per day or time.	

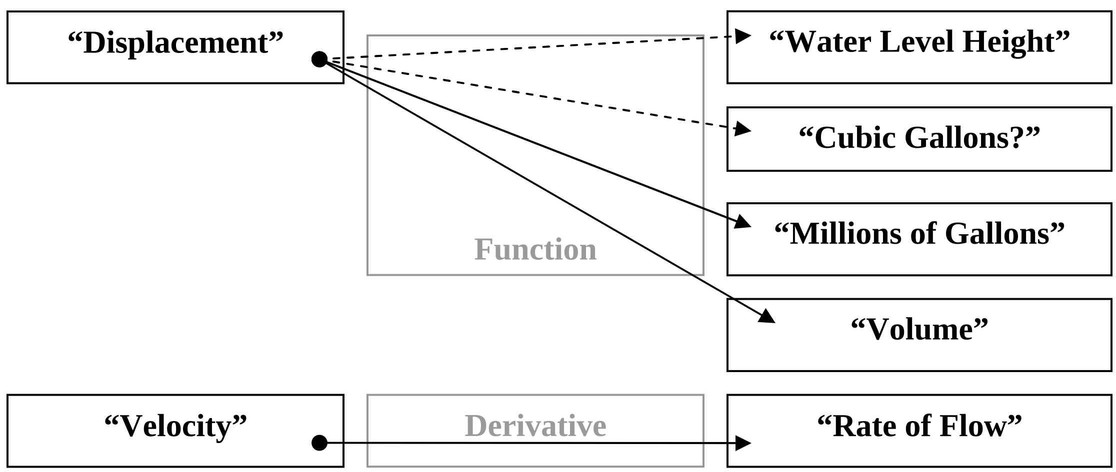


Figure 18. Justin's attempts to map "velocity" and "displacement" to vocabulary in the Quabbin Reservoir context.

At this point (clip 622) Justin erases his second written label for the volume graph, "volume," and writes "millions of gallons." He does this with hesitation, verbally inserting the term "cubic gallons" as he searches for the correct label. Both Daniel and Justin verbally add the term "per day," although Justin does not write the incorrect "per day" on his volume graph. Daniel, on the other hand, may have correctly added the "per day" term, because his next statement seems to indicate that he is actually looking at the rate graph.

627	(0:48:29.9)	Daniel:	Then our, or that's the uh derivative, that's the volume, yeah, that's the, what'd we call that?
628	(0:48:40.5)	Daniel:	What do we call the rate of change of the volume? Is that what we're calling our derivative?
629	(0:48:46.8)	Justin:	No it'd [*the derivative*] just be the rate of change of, it wouldn't be rate of change of the volume, um, it would just be-
630		Daniel:	Rate of, of flow?
631		Justin:	-it [*the derivative*] would be rate of, it would be rate of change of the-
633	(0:48:58.0)	Daniel:	Of water?
634	(0:48:58.5)	Justin:	Yeah of the millions of gallons.

Surprisingly, Justin says that the derivative wouldn't be the rate of change of *volume*, but that the derivative would be the rate of change of the *millions of gallons*. Justin has previously recognized that "millions of gallons" implies a volume measurement, and so it is not immediately apparent why he feels that one term would be more appropriate than the other. Figure 19 shows Justin's process of determining a name for the derivative of volume. Where he has accepted the terms "volume" and "millions of gallons" as analogs for "displacement," Justin has also just stated that the derivative would not represent "change in volume," but "change in millions of gallons" instead.

Upon hearing Justin's comment, Jamie contradicts Justin's statement that it "wouldn't be the rate of change of volume," by stating that the rate of change of volume is the same thing as the rate of change of the millions of gallons. Daniel (after some hesitation) and Justin agree. Justin goes on to justify Jamie's statement by referring to a familiar context for studying derivatives, the context of motion.

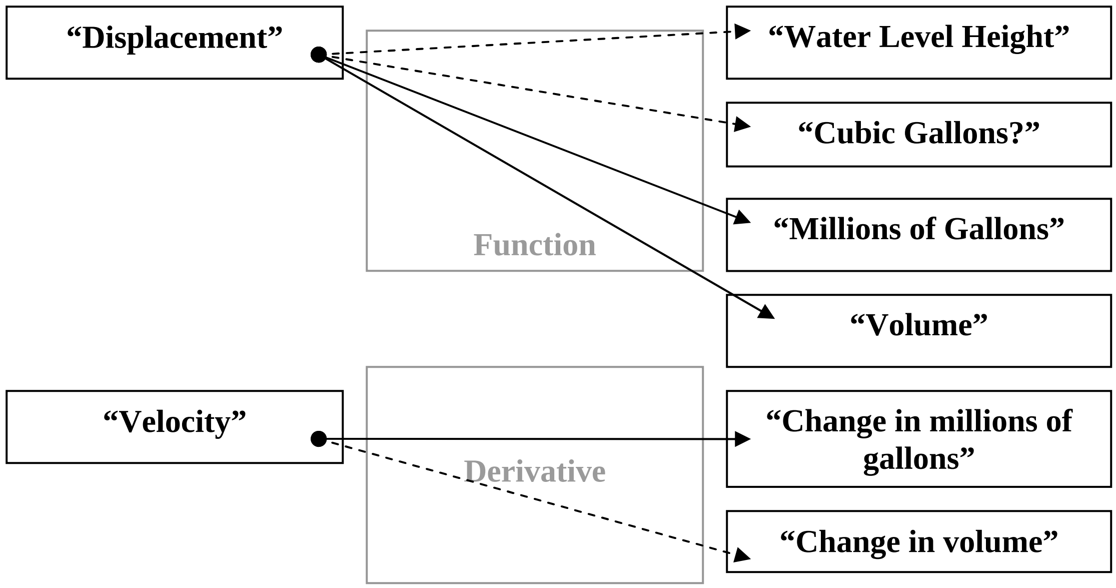


Figure 19. Determining a name for the derivative of volume.

635	(0:49:00.4)	Jamie:	The same thing as rate of change of, our volume.	
636		Justin:	Huh?	
637	(0:49:03.6)	Jamie:	That’s [*the rate of change of the millions of gallons*] the same thing as the rate of change of our volume.	
638	(0:49:05.7)	Justin:	Yeah.	
639	(0:49:09.1)	Daniel:	Would it? [short pause] Yeah it should be-	
640	(0:49:12.2)	Jamie:	Yeah.	
641		Justin:	Yeah.	
642	(0:49:12.5)	Daniel:	-because if you’re taking dy dx it’s the rate of-	[writing on his paper]
643	(0:49:14.4)	Justin:	‘Cause it’d be like using meters and saying your rate, your change in distance. So it’d be change in volume.	
644	(0:49:19.4)	Daniel:	[sighs]	[writing on his paper]
645		Justin:	We’re working gallons and so it’d [*our derivative*] be change in volume.	
646		Jamie:	Yeah.	

647	(0:49:22.6)	Justin:	If you were working in meters it'd [*the derivative*] be your change in distance.	
648	(0:49:24.6)	Jamie:	Uh huh.	[writing on her paper]
649	(0:49:24.9)	Justin:	And so it'd be your distance graph. So this would be considered our volume graph.	[touching his volume graph] [Justin's volume graph]
650	(0:49:28.1)	Jamie:	Yeah.	
651	(0:49:28.4)	Daniel:	Yeah. I like that. The change of [3 sec] the change of [1 sec] VOLUME with respect to time.	

In this final portion of transcript, Justin lays out an additional element of his analogical language framework: the units that are used to label a function and its derivative in the velocity and volume contexts (Figure 20).

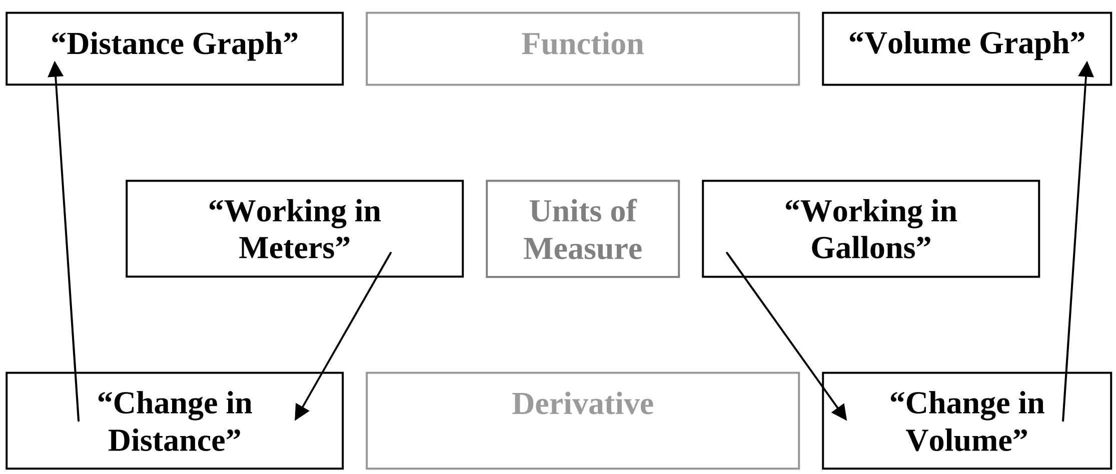


Figure 20. Justin's fitting of units of measure into the analogical language structure.

Justin speaks of "using" and "working in" specific units of measure. In the velocity and displacement context, he speaks of "using" meters as a unit to measure distance. In such a case, the derivative would be more naturally described as representative of "change in distance" rather than "change in meters." For volume, he speaks of "using" gallons to measure volume. Reasoning analogically, Justin concludes that the derivative would be said to show "change in volume" rather than "change in millions of gallons." This further settles Justin's hesitation as to what to actually write as a label for his graph of quantity. He reasons that in a situation where "you were working in meters . . . it'd be your distance graph," and therefore, in their situation of "working in gallons" they would name their graph a "volume graph."

Having sorted out subtle differences in the meaning of "volume" and "gallons," Justin finally labels his graph with both terms. He places "volume" as a general title for the graph, and "millions of gallons" next to the vertical axis, indicating the units in which values on the vertical axis are measured (Figure 21). Daniel also seems content with the title they have found for the "derivative of volume," which is "the change of volume with respect to time" (651). With a resounding "Woohoo" from Daniel, the group moves on to tackle part (b) of the Quabbin Reservoir Task.

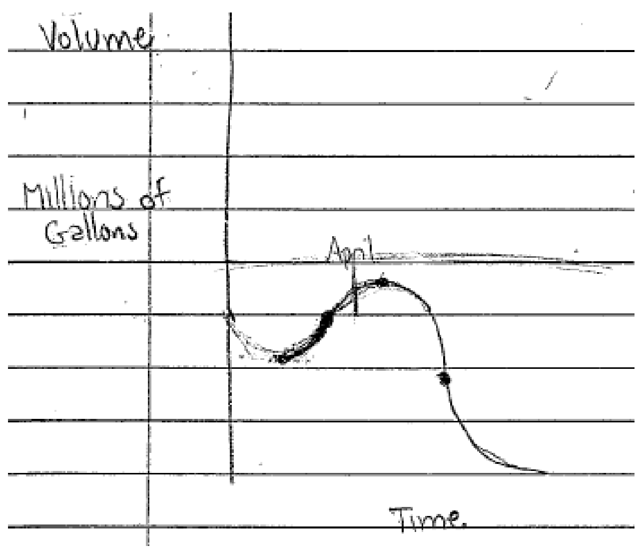


Figure 21. Justin's volume graph with "volume" as a title and "millions of gallons" as units for the vertical axis.

This last except of transcript may help to illustrate a possible weakness of viewing the learning of mathematics strictly from the perspective of becoming a participant in discourse. This weakness has to do with how discourse as a social practice may serve to mask the contributions, or even existence, of individual cognition. As Justin has played a definite role in his group as one who listens, explains, organizes, and justifies, the result has been that I, as an observer, can say very little about how *he* has come to believe that what he is saying is true. For the most part, the analysis here may suggest that vocal egocentric speech does not exist in the language of university students. However, this final portion of transcript (643-651) appears to be an exception that suggests that egocentric speech does exist and may, in fact, be viewed as a means for evaluating individual cognition.

Although his analogical argument (643-651) may serve as a response to the questions of his peers, Justin's language also suggests that the primary question that he is answering may be his own. With considerably less hesitation than in previous portions of transcript, Justin now speaks as though resolute. While his earlier explanations, specifically in Segment 9: The Gospel According to Justin, were constantly interrupted by his characteristic, "Does that make sense?" Justin does not stop to check for understanding or agreement. This may be because Justin, in clips 643-651, is either negotiating language with himself, or using language to express relationships that he has recently come to know. He is answering *his own* questions about the analogical relationships and language that are central to *his* mathematical understanding. Because Justin's language does not include the cues that generally reflect his efforts to place himself at the point of view of the hearers of his speech, I view this portion of transcript as an example of more egocentric speech that allows me as an observer, to make conjectures about the structure of Justin's individual thoughts. As has been explained earlier, due to the concern that the participants often expressed for *one another's* mathematical understandings, egocentric speech as evidence to support such conjectures is relatively rare to this data. Therefore, while I have said much about the social negotiation of mathematical meaning and language, it is quite possible that the role of individual cognition may have taken a secondary role in this analysis. The resulting implications are discussed in the next chapter.

In this chapter, I have presented three narrative strands of the negotiation of meaning and language in mathematical discourse. To negotiate meaning for the conventional terms "point of inflection" and "concavity," Daniel used personal pronouns and personal experience to convey his meaning for "point of inflection," which was later connected to Justin's meaning for "concavity," resulting in a view of inflection points that functioned to help Justin to relate the graphs of a function and its derivative. In the negotiation of language for Daniel's construct of "zero points," Jamie, Daniel, and Justin purposefully chose language that consistently reflected their personal experience to assist their interpretation of mathematical concepts. Finally, Justin's analogical reasoning process was revealed as the participants pursued answers to each others' questions and comments about "velocity," "volume," and "the derivative of volume."

CHAPTER 7: DISCUSSION

In defining human agency, Inden (1990) stated that when making a choice, human agents "may consider different courses of action possible and desirable, though not necessarily from the same point of view" (p. 23). The analysis of three processes of negotiation of meaning presented here highlights how different points of view, or different explanatory factors, may be reflected in the participants' choice of language. In this chapter, I discuss situations in which the explanatory factors of one's experience and one's understanding of the mathematics can lead to improvisation or compromise in mathematical discourse. Looking at conventional mathematical language as an abstraction of the analogous features of various contexts and personal experiences, I suggest a Mathematical Language Matrix as a way to characterize mathematical language. I also suggest implications for the teaching of mathematics "in context."

The participants in this study have displayed more than just a variety of mathematical language. Their choices have also demonstrated how mathematical discourse truly is a social practice. Each participant had different experiences and expectations to contribute to mathematical discourse, and each exhibited a "mindful awareness of the impact one's actions and choices may have on others" (Walter & Gerson, 2007, p. 209). The third explanatory variable for choices made in mathematical discourse, that of the social situation in which the discourse occurs, was reflected in Justin's and Daniel's use of social speech that may be said to epitomize Piaget's description of "attempting to place himself at the point of view of his hearer" (1997/1896, p. 9). As this type of speech may be used to increase students' access to mathematical discourse, instructors would be wise to, after Lampert (1990), model, describe, and negotiate (Cobb, Wood & Yackel, 1993) similar types of speech in their mathematics classroom.

Finally, it should also be noted that the mathematical discourse described here emerged as a result of the participants' choices as they developed their own solution to the Quabbin Reservoir Task. Their participation in mathematical discourse wasn't just a process of taking on roles in some pre-existing conversation, and stands in drastic contrast to notions of scripted mathematical discourse. In a sense, the term "participation," may not completely capture the contributions that Daniel, Jamie, Justin, and Julie made. Through the exercise of personal agency, they were the creators, moderators, and evaluators of personally and socially meaningful mathematical discourse. Indeed, the viewpoint of "becoming a participant in mathematical discourse" may be limited not only by traditional views of mathematical discourse, but also traditional views of participation. In the sections that follow, I synthesize my observations and analyses of the mathematical discourse of these participants to define mathematical discourse from the perspective of personal agency. This definition may help researchers to broaden and refine their notions of mathematical discourse, and help teachers to appreciate the value of student language and thought.

Mathematics and Experience

In choosing his mathematical language, we have seen how Justin was careful to not only be mathematically correct, but also experientially correct. That is, Justin avoided making interpretations of the mathematics that couldn't be mapped to personal experience. His avoidance of a net rate interpretation of a zero point in terms of physical phenomena reminds me of the choice of a practicing elementary school teacher named

Matt when he was completing a similar task about water in a reservoir (Johnson, 2005; Walter & Johnson, 2007). This teacher relocated the context of the task to a bathtub, where the bather specifically controlled inflow and outflow, either by turning on the faucet, turning off the faucet, opening the drain, or closing the drain. Simplifying the task by not allowing water to be entering and exiting the bathtub simultaneously, Matt's consequential interpretation of a point where the net rate graph crossed from the upper half plane to the lower half plane was as follows:

> Then we think, 'you know what, I've got enough water,' so we start turning the knob off at this point . . . as we're turning we're decreasing the gallons a minute from one and half gallons per minute to one gallon per minute to zero gallons per minute right here. Then we notice we've got too much, **so immediately, no time between when we turn it off and when we start turning the other knob to let the water out,** we notice we have too much, so we start to turn the knob at this point to let the water out, and we're slowly turning that knob, or slowly opening the drain . . . (Johnson, 2005, p. 56, emphasis added)

This elementary school teacher, in inventing a situation that may be explained by the given graph of rate, presented a situation that may be slightly unrealistic. With absolutely no time passing, the bather turns off the inflow at the very moment that he turns on the outflow, resulting in a virtual "point" at which the net rate of change is zero. This may be considered a case where personal experience is momentarily improvised in order to correctly reflect the mathematics of the situation.

In the current data, the opposite phenomenon might be observed, in which the mathematics is momentarily compromised in order to allow for language that fits personal experience. When the participants spoke of a zero point as a "period of no change," and found alternatives to describing net rates of change as physical combinations of inflow and outflow, they may have danced around some of the more technical details of the mathematics in favor of preserving the semantic nature of their argument. In both cases, however, the momentary compromises were acceptable to the listeners. This suggests that there may be an underlying unspoken assumption that relationships between mathematical events and personal experience are not to be interpreted in a strictly literal sense. Rather, the participants play "the believing game" (Elbow, 1973) in order to extract the more general sense of the comparison rather than get caught up in little details.

Organizing Conventional and Ordinary Language

At the beginning of this study, I described a continuum of conventional and mathematical vocabulary, characterizing conventional mathematics language as language having precise and abstract meanings in mathematical discourse, such as "derivative." I hypothesized that every conventional mathematics term could also be described in ordinary language, such as Justin's description of velocity as the derivative of displacement as "how fast you're changing your distance." Figure 22 represents my initial ideas about the placement of the term "derivative" and the phrase "how fast you're changing your distance" on this continuum of mathematical language.

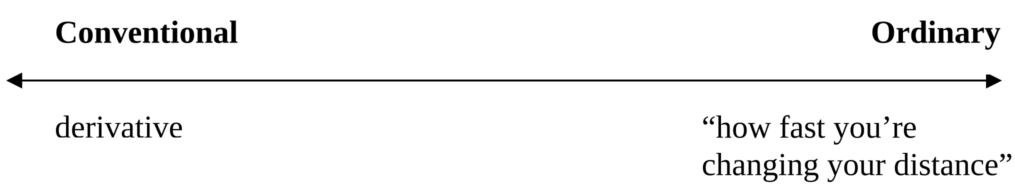


Figure 22. A continuum for mathematical language.

The data were also rich with mathematical language that I could not characterize as completely conventional mathematical language, nor completely ordinary. Some examples would be "instantaneous rate of change," and "velocity." I imagined these two terms as located at some point along the mathematical language continuum between the extremes of conventional and ordinary (Figure 23). The curved arrows cycling between "instantaneous rate of change" and "velocity" in Figure 23 represent my confusion about which I considered to be "more" conventional. I considered "velocity" a precise conventional term, but not a strictly mathematical term because of its importance in physics. "Instantaneous rate of change" didn't seem to belong to any specific application of mathematics, and was therefore very abstract, yet not exactly as precise as "derivative" or even "velocity." In fact, "instantaneous rate of change was more of a description or interpretation of a derivative.

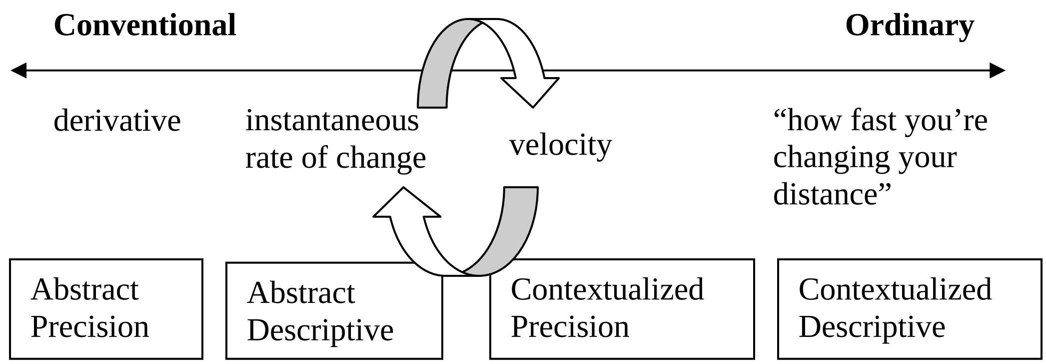


Figure 23. Completing the continuum for mathematical language.

As I attempted to place various types of language along my continuum, I continued to encounter the same phenomena. Some language was very conventional in that it was abstract and precise; other language was abstract, but more descriptive than precise; and other language was precise, but less abstract because it was strongly associated with a particular science or context other than mathematics. I concluded that I would either have to make a choice between abstraction and precision as the more salient feature of conventional mathematics vocabulary, or change the structure of my continuum. I decided to change the structure of my continuum, creating a two dimensional array for characterizing mathematical language. This array, which I named a Matrix for Mathematical Language, essentially combined two continua, one represented by the extremes of precision and description, and the other by the extremes of abstraction and contextualization (Figure 24). Precise and abstract mathematical language could now be found in the upper left hand corner of the matrix. Descriptive but abstract terminology, such as mathematical definitions, descriptions and abstract interpretations, was located in the upper right hand corner. Contextualized analogs or examples of precise mathematical terms found a place in the lower left hand corner of matrix, and contextualized descriptions found place in the lower right hand corner.

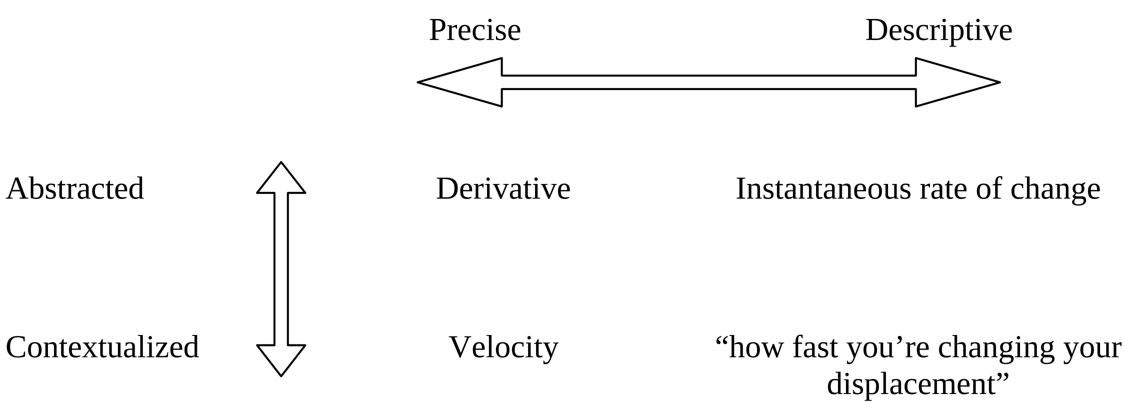


Figure 24. Matrix for mathematical language

I realized that this matrix could be extended indefinitely, as new contexts, such as the Quabbin Reservoir context, could be added. I didn't know if I could claim that one context was more "contextualized" than another. Furthermore, I reasoned that mathematics itself *was* a context, the context of abstraction. I changed my representation once again, replacing the continuum of more or less contextualized language, with a list of different contexts. I then added the Quabbin Reservoir Context to this list (Figure 25).

Context	Precise ⟵	⟶ Descriptive
Mathematics	Derivative	Instantaneous rate of change
Kinematics	Velocity	"how fast you're changing your displacement"
Quabbin Reservoir	Rate of flow	"how fast you're changing your volume"

Figure 25. Extended matrix for mathematical language.

Learning to Think Mathematically

Although the participants may not have literally organized the language of their mathematical discourse in a matrix as I have, the data suggest that they did demonstrate an inclination to organize language by identifying analogs for mathematical concepts in different contexts and interchanging precise and descriptive language for the purposes of explanation and justification. These organizational habits may be reflective of an ongoing process of learning to think mathematically that takes place as students engage in discourse with the goals of solving mathematical problems and advancing their own mathematical understanding. Not only did the participants find and agree upon a solution to the given task, but they engaged in the mathematical processes of property noticing, interpretation, conceptualization, abstraction, and generalization.

Like Daniel, I was not entirely satisfied with the analog of derivative in the Quabbin Reservoir Context as "rate of flow." As mentioned by Zandieh (2002), the language of derivatives in the field of kinematics is highly developed, as “velocity,” “acceleration,” and even “jerk” (the time derivative of acceleration) are quantities that are measured and studied independently. Zandieh suggested that the precise language structure is a motivation for studying derivatives in kinematic contexts. Attempting to fill in the matrix for derivative in the context of the Quabbin Reservoir reveals that there is no commonly known “special name” for “the derivative of volume.” As mentioned in Chapter 6, the net rate of change of volume, or derivative of volume, is not generally measured in a reservoir. Inflow and outflow, on the other hand, not only exist in the world of mathematics, but are also observed and measured as quantities in fields such as engineering, agriculture, and, as reported by Walter and Johnson (2007), the more common human experience of filling up a bathtub. The design of the Quabbin Reservoir Task (Hughes-Hallett et al., 1994) accurately reflects this fact by providing information about inflow and outflow, rather than combining the data as a net flow.

Sfard (1997) suggested that metaphors are used for conceptualizing, while, analogies are used for reasoning. The matrix of mathematical language may provide an example of Sfard’s notion of metaphor. When Daniel asks, “What would the derivative of volume be?” his demonstrated meaning for mathematical language may be represented by Figure 26. Reasoning that the derivative of displacement, velocity, is a quantity that can also be measured, Daniel conceptualizes the idea of a derivative of volume. Although he has not been told that a derivative of volume exists, and does not know what such a derivative would be named, he reasons a derivative of volume into existence. This derivative of volume, which did not previously exist in Daniel’s language matrix, comes into existence through the vehicle of metaphor.

Context	Precise	Descriptive
Mathematics	Derivative	Instantaneous rate of change
Kinematics	Velocity	"how fast you're changing your displacement"
Quabbin Reservoir	*"What would the derivative of volume be?"*	"how fast you're changing your volume"

Figure 26. Daniel's matrix for mathematical language for the concept of *derivative.*

One possible argument for contextualized mathematics instruction can be formed by considering the impact of adding additional contexts to the language matrix in Figure 25. Each time a new context is added, more language becomes relevant to the study of mathematics, giving students opportunity to build upon and refine their meaning for new types of language as they learn to engage in the mathematical activity of identifying and abstracting patterns from a variety of contexts. A completely abstract approach to mathematics limits students' opportunities to build meaning and language, especially when the abstract language of mathematics does not intersect often or well with students' ordinary language. The addition of contexts to mathematics instruction implies an addition of language, and if the language of these contexts intersects appropriately with students' experience, this language offers additional pathways for students to access mathematical discourse. Furthermore, students are then given rich opportunities to participate in the process of abstraction as well as operate on the results this process. If, however, additional contexts are not reflective of, or closely related to, student experience, the addition of these inappropriate contexts may further impede students' access to both the discourse and the mathematics (Zevenbergen, 2000).

Another possible danger in contextualizing mathematics activity is that teachers may take the extreme stance of completely eliminating the abstract language of the mathematical context because it may seem irrelevant or impractical. However, Brown (2001) recognized the teaching and modeling of conventional terminology as a necessary role of the mathematics teacher, along with providing students opportunities to use language that symbolizes their own experience. Sfard (2000) also suggests that what some may call the premature introduction of language may in fact be a necessary step in the process of learning about mathematical concepts. When Daniel introduced the term “inflection point” to discourse, Julie and Jamie’s questions revealed that they had not previously engaged in extensive negotiation of meaning for the term. Although Jamie could recognize an inflection point as a maximum or minimum on a derivative graph, she had not yet noticed how points of inflection were also represented in the shape of the original function. Daniel’s introduction of the conventional language of inflection point motivated the development of descriptive language from a variety of contexts to create and negotiate meaning for a mathematical idea that may not have been naturally encountered in the contextualized realm (Figure 27). As a result, the participants learned to notice new mathematical properties of mathematical objects.

Figure 28 represents a possible mathematical language matrix for the concept of ”zero points.” The term, “zero points” is not a conventional mathematics term, nor does it suggest the context of velocity or water in a reservoir. To represent the nature of Daniel’s coined term “zero points,” I introduce a new context for language that is descriptive of the shape of a graph, with “zero points” being a more precise description of “where the graph crosses the horizontal axis.” As in Figure 27, the matrix for *inflection points,* it is useful to identify individual frames of reference that may be represented by different graphs within the various contexts. In this matrix, I have also included the name of the participant who introduced the various descriptive terms to discourse. Each of the participants offered different ways of describing zero points, thus creating a richer definition and interpretation for Daniel's coined term. In doing so, they developed ways to apply the shape of an abstract graph towards the purpose of interpretation of physical phenomena in the rich context of water in a reservoir.

Context (Frame of Reference within a context)	Precise ⟷	Descriptive
Mathematics (Function)	"Inflection Point"	"Where concave down changes to concave up"
Mathematics (Derivative)	Extrema	"Where the slope is the highest"
Kinematics		"Where the velocity is the highest on a displacement graph"
"A Slide"		"The point where you start to level off"

Figure 27. A co-constructed mathematical language matrix for "inflection point."

The findings of this study suggest that, to assist students in learning to think mathematically, the purpose of studying mathematics in context should be made explicit. Situated cognition views of learning (Brown, Collins, & Duguid, 1989) suggest that students may compartmentalize their activity according to context. For example, a student in a contextualized calculus course may report that they learned about velocity one week and rate of flow the next week, but never make the abstract connection between the two contexts which a mathematician might refer to as "derivative." Students may need to be explicitly taught that mathematical concepts are intended to be abstract concepts that can be applied to and observed in various other contexts. Therefore, the primary purpose of studying mathematics in context is not necessarily to give mathematics a more familiar appearance, but to provide opportunities for students to learn to identify, compare, and abstract mathematical concepts from these contexts. Analogical problem solving may help students to participate in these activities, but if mappings and solution processes are not made explicit, students may find themselves overwhelmed by a large amount of language with few connections in meaning for that language.

Context (Frame of Reference within a context)	Precise	Descriptive
Mathematics (Function)	Extrema	"Maximums and minimums" (Justin)
Mathematics (Derivative)	Critical Points[5]	Zeros of the derivative
Shape of a Graph (Rate)	"Zero Points" (Daniel)	"Where [the graph] crosses the x-axis" (Justin and Daniel)
Shape of a Graph (Original)		"Where they meet" (Daniel)
Shape of a Graph (Quantity)		"Your top and bottom points" (Julie)
Kinematics		"Where the velocity (of the water) is zero" (Daniel)
Quabbin Reservoir (Separate Rates of change)		"When the inflow equals the outflow" (Jamie)
Quabbin Reservoir (Volume)		"When the volume of the water isn't changing" (Justin)
Quabbin Reservoir (Net Rate)		"When the net flow is zero" (Justin)

Figure 28. Mathematical language matrix for "zero points."

[5] In a strict mathematical sense, critical points and extrema do not refer to the same concept. However, for the purposes of our participants in completing and explaining the Quabbin Reservoir Task, these concepts coincided well.

Instructors and students might be encouraged to make their analogical language explicit by identifying their analogical language use. To further capitalize on the abundance of mathematical language, students and instructors might even create their own matrices of mathematical language as they participate in contextualized mathematical activity. As demonstrated by the choices of the participants in this study, both instructors and students have the potential to ask questions and make comments that encourage the negotiation of meaning and language toward making relationships between different types of language more explicit.

The Values of Mathematical Discourse

If defining types of discourse is to go beyond word choice and content to include Gee's (1996) idea of "ways of behaving, interacting, [and] valuing" (p. viii), the choices of the participants in this study may provide a basis on which we may begin to define mathematical discourse. Julie leads the way, pressing her peers to make their analogical reasoning, solution processes, and language explicit with summarizing statements and clarifying questions. These questions encouraged the other group members to share and revise their thinking and language through engagement in the negotiation of meaning.

Jamie exemplified the mathematical value of precision. In originally coding the data for pronoun use, I noticed that statements by Daniel, Julie, or Justin, were replete with the pronouns such as "this" and "that" and "it." Jamie's language was exceptional in this respect. I noticed that I didn't need to hypothesize about possible referents for her pronouns because she rarely used these pronouns at all, and made the referents of the pronouns she did use explicit through gestures and pointing. Jamie sought precision in her language, and was conscious of the different labels that were being used for different graphs. The transcript below demonstrates Jamie's refusal to accept Daniel's misuse of the term "midpoint," which encouraged Daniel to add even more labels to the horizontal axis of his graph so that he could be more precise as he referred to different portions of the graph. The value of the labels was made apparent to me as I used them to refer to different portions of the graphs in my process of analysis.

825	(0:56:41.9)	Daniel:	And after February, or after the midpoint, of between January and April-
826	(0:56:45.3)	Jamie:	What is the midpoint? I don't think it's, like the middle of February?
827		Daniel:	it's not February, but-
828	(0:56:49.9)	Jamie:	Okay.
829	(0:56:50.4)	Daniel:	I'm just saying like-
830		Jamie:	The midpoint.
831	(0:56:51.8)	Daniel:	-the midpoint.
832	(0:56:52.5)	Jamie:	Okay, I understand.
833		Daniel:	I don't know [inaudible]
834	(0:56:54.1)	Daniel:	Ooh. We could put like "A" and "B," and "C" [labeling the horizontal axis with the letters A through G].
835		Jamie:	Yeah.
836	(0:56:57.3)	Daniel:	Wooh! I 'm so excited!
837	(0:56:59.4)	Jamie:	[laughs]
838	(0:57:00.2)	Daniel:	Yes! I got all the way up to "G"!

Daniel's tendency to search out and even create labels for important mathematical objects suggests a view of the role of personal agency in the hermeneutic cycle of mathematical language. Daniel was not afraid to act creatively on language as he coined phrases and created examples, but at the same time he exhibited awareness of a larger system of conventional language and ideas. Daniel also exemplified the value of sense making. Viewing mathematical language and concepts as part of a structured system, he built on his knowledge of the relationship between displacement and velocity to conceptualize and reason about the relationship between rate of flow and volume. Daniel even searched for language to complete his mapping of the two problem solving contexts.

Combining creativity with sense making, Daniel melded the idea of maximum velocity with his personal experience to develop a kinesthetic description of a point of inflection. Daniel also recognized mathematics as a social practice as well as a personal practice, and was always the first to check whether his peers felt comfortable with his point of view with questions such as, "Is everyone good with that?"

Justin also developed a habit of checking with his peers, repeatedly asking, "Does that make sense?" and waiting for a response before moving on. Justin demonstrated an acute awareness of what his peers were, and had been, saying. He based his decisions for language not only on his personal experience and understanding of the mathematics, but also his best approximation of his peers' understanding of the mathematics. One of the ironies of Segment 9: The Gospel According to Justin is the fact that it cannot be classified as strictly "according to Justin." Analysis has shown how many of Justin's decisions about language reflected the language of his peers in previous discourse. From his use of "negativity" after Daniel, to his care to say that the rates are "becoming equal" rather than "coming to zero" as suggested by Jamie, to his qualification of the metaphorical term "velocity" to show his respect for Julie's preferences, to his final triumph in organizing two different terms for quantity of water as prompted by his instructor, Justin's words provide evidence of one of the most important processes in the negotiation of meaning: listening. Although Justin may have been able to correctly explain his solution using his own preferred language, his language awareness demonstrates an acute awareness of his peers and a level of social speech that goes beyond simply acknowledging the presence of his listeners to actually speaking the words that they have spoken.

New Forms and Functions for Social Speech

My application of Piaget's (1997/1896) notion of social and egocentric speech suggests possible characterizations of the roles of social and egocentric speech in mathematical discourse. To characterize social speech, I searched for evidence of the speaker attempting to place him or herself at the point of view of the hearer, which resulted in the identification of some specific uses of pronouns and the dialogic nature of my participants' speech. For example, in the "inflection" narrative, Daniel used the second person pronoun "you," not to accuse or command, but to invite Julie to reflect upon her own personal experience. Daniel's language reflects his purposeful choice based on his meaning for point of inflection and his meaning for Julie as one who may relate to the experience of riding down a slide.

142	(0:19:19.1)	Daniel:	Okay, so if you're, like, here's kind of an, idea, okay. So if you're drawing, a curve. Um, like, you just. Ah.
143	(0:19:32.9)	Daniel:	Okay, so the inflection point is where the velocity is the highest,
144	(0:19:37.3)	Daniel:	so, like, if it, if you were like going on a slide and if you're falling down on it . . .

I found that language not only may reveal the choices that the participants make, but also the choices that they do not make. For example, Daniel's hesitation (142) demonstrates that, although he considered approaching his explanation from the perspective of drawing a curve (an explanation that he does later offer), he opted for an experiential example in which his own understanding of inflection as point of "highest velocity" can be interpreted quite literally. The three explanatory factors for language in mathematical discourse are reflected in Daniel's choice. Daniel's language reflects his own mathematical understanding of points of inflection as points of highest velocity. The notion of riding down a slide draws on Daniel's personal experience. The social implications of Daniel's concern and respect for Julie as a member of their collaborative group may have encouraged Daniel to bring in the factor of personal experience as a frame for his explanation.

Another example of social speech as "placing [oneself] at the point of view of the hearer" was demonstrated in Justin's habit of revoicing, which may be viewed as the practice of adopting and adapting the *language* of the hearer. Although he demonstrated the capacity to explain the mathematics of the Quabbin Reservoir Task using various types of language, Justin chose to demonstrate appropriate use of the language of his fellow participants. For example, in Segment 9, we see Justin echo a previous conversation of Jamie and Daniel, stating that, instead of talking about net rate "coming to zero," he chooses to say "they're gradually becoming equal" as was suggested by Jamie. We also see that Justin is not merely parroting Jamie's language, but that he appropriately indicates on the original graph the point where the inflow and outflow curves intersect.

352	(0:33:20.9)	Justin:	So it's lower right here. Now it's [*the rate graph*] gradually coming to zero,	[the rate graph at A] [the first intersection of the inflow and outflow on the original graph]
			it's gradually, um, coming to, it's gradually, **I don't want to say coming to zero**, I wanna say, [2 sec]	[the first x-intercept on Justin's net rate graph]
			yeah they're [*inflow and outflow*] gradually becoming equal, so it's the water level,	[the first intersection of the inflow and outflow on the original graph]
			the volume level is staying the same. Right?	[holding hand flat with palm down at eye level]

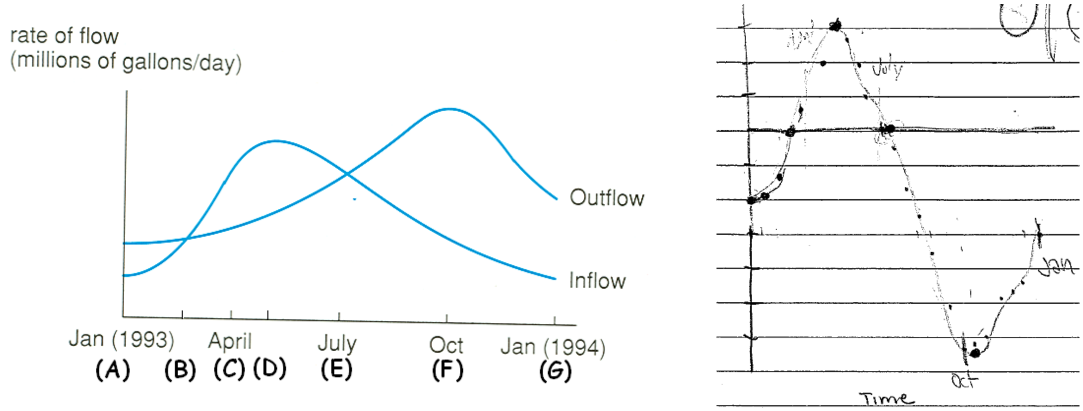


Figure 29. The original graph (left) and Justin's rate graph (right).

Speiser (2002) demonstrated how even young students are capable of "trying on" the thinking of others, and inviting others to do the same. These invitations to participate in discourse are structured in the present data by Daniel's use of examples and pronouns and Justin's revoicings. The result is a type of social speech that goes beyond Piaget's categories for young children. Like O'Connor and Michael's (1996) characterization of revoicing, this social speech functioned to invite and allow all participants access to mathematical discourse. A fellow researcher pointed out that, without this form of social speech as a norm in mathematical discourse, Julie's important contributions to the negotiation of meaning may never have entered the social discursive stage.

Egocentric Speech and Implications

As mentioned earlier, a possible weakness in the methodology and analysis of this study may be the way in which individual cognition may be obscured or ignored when mathematical activity is viewed through the lens of the social practice of mathematical discourse. If social speech, as discussed earlier, may be viewed as the participants' efforts to adopt and understand one another's language and points of view for the purpose of negotiating mathematical language and meaning, egocentric speech may continue to represent the other end of the continuum. That is, egocentric speech would be speech in which the participants do not make an effort to adopt one another's language or point of view, and therefore, the participants use their own language for their own purposes. They vocalize their inner thoughts, not for the purpose of explaining or even communicating with others, but towards the ends of developing, clarifying, or solidifying their own mathematical understandings.

Piaget (1997/1896) noted that egocentric speech is not intended to serve social functions, and as a result is often not observed in the language of adults in social situations. Therefore, a researcher or mathematics teacher who is interested in accessing the individual cognition dimension of mathematics learning with as little interference as possible may choose to ask learners of mathematics to directly explain their mathematical understandings, or perhaps less directly, to explain why they made the decisions that they did when solving mathematical problems. If such questions are interpreted by learners as invitations to participate in egocentric speech, it may be possible to achieve a better approximation of how those individuals' mathematical understandings are developing. After listening carefully to egocentric explanations, the researcher or teacher may participate more genuinely in social speech for the negotiation of meaning.

As an example, the instructor-participant in this study exhibited sincere questions as she asked the participants to explain the intended meanings for their language (Daniel's "zero points") and gestures (Justin's rising and falling palm for "water level"), as shown in the transcripts below. In response to the instructor's questions, the participants reflected on and eventually expressed their reasons for their choice in language and gestures, providing further evidence that allowed me as a researcher to provide more convincing interpretations of the participants' individual thoughts.

208	(0:23:34.3)	Dr. Walter:	I don't understand **what you mean**, where the inflow and the outflow meet you're gonna have zero points. Zero points of what?	
209	(0:23:42.7)	Daniel:	The level of water overall. So the velocity. **I think, let's see.**	
220	(0:24:22.5)	Daniel:	And so, when the, when there is no change in the water level for a certain time, the **velocity** will be zero . . .	[holding both hands at the same level]

284	(0:27:36.4)	Dr. Walter:	So you're thinking of measuring the quantity of water in the reservoir by the height [1 sec] of water in the reservoir?	
			When you're doing this I'm imagining you're talking about the height?	[raising and lowering flat hand]
286	(0:27:48.2)	Justin:	That's how I, that's how **I** think about it, cause I don't know how else, I guess you could measure it in, like volume, but,	
			I don't know, height, just, **to me**, seems more, one, two dimensional.	[raising and lowering hand with palm down]

The egocentric reflection and speech that resulted from such questions not only helped Daniel and Justin to clarify and better their own mathematical understandings, but these questions were also seen as initiatory to extended processes of social negotiations of meaning. As has been demonstrated by the findings of this study, an in-depth analysis of the negotiation of meaning and language in mathematical discourse can suggest new and complex ways of viewing the process of learning mathematics. However, as recognized by Sfard (2001), such findings are to be viewed as *interpretations* of the participants' *intentions*, and are not to be viewed as absolute truth or facts. Here I suggest an additional limitation to such studies, being that, without specific invitations for the participants to participate in egocentric speech, the continuous influence of the participants' concern for how their choices may affect one another, may obscure and possibly even eliminate the importance of individual mathematical thought. However, just as students and teachers can use revoicing and other forms of social speech to emphasize the role of collective understanding mathematical discourse, I would suggest the participants in discourse also have the capacity to ask questions and make other decisions that shape discourse in a way that also emphasizes the existence and importance of individual thought in social practices.

Defining Mathematical Discourse via Agency

Finally, it should be noted that the perspective of learning mathematics as becoming a participant in discourse also has the potential of obscuring the characteristics that make mathematical discourse mathematical. In this study, I have demonstrated how three factors might be viewed as explanatory for choices made in mathematical discourse. Applying Walter and Gerson's (2007) definition of personal agency as the "requirement, responsibility and freedom to choose based on prior experiences and imagination, with concern not only for one's own understandings of mathematics, but with mindful awareness of the impact one's actions and choices may have on others" (p. 209), I have discussed how "prior experiences and imagination" are reflected in students' decisions to use analogy and analogical language as vehicles for mathematical conceptualization and reasoning. While relating mathematics to personal experience, the students have carefully negotiated meaning that not only matches previous experience, but also reflects their own efforts to communicate and improve their own mathematical understandings. The participants in this study have also exhibited "mindful awareness of the impact one's actions and choices may have on others," by developing forms of social speech that may increase participants' access to discourse.

At the beginning of Chapter 2, I cited Goodwin's (2000) suggestion that the ideal context for the study of human cognition, language, and action is a situation where participants carry out action through talk. Goodwin stated that these participants should not, however, be placed in sterile clinical environments for such studies, but be simultaneously attending to "larger activities that their current actions are embedded within," and "relevant phenomena in their surround." I now suggest an analogical language mapping of my own. I view "larger activities that their current actions embedded within," as analogous to the social sphere in which mathematical discourse takes place. I also view "relevant phenomena" as analogous to the experiences, linguistic and otherwise, which the participants bring to mathematical discourse.

The third explanatory factor for human choice in mathematical discourse, "concern for one's own understanding of the mathematics," does not have an analog in Goodwin's description. This is not entirely surprising, as Goodwin was not specifying his work to mathematics education. My resulting suggestion, then, is that the third explanatory factor may be what makes mathematical discourse mathematical. While the exercise of personal agency in all forms of discourse may reflect the social and experiential explanatory factors, not all discourse reflects the participants' concern for their understanding of the relevant mathematics. At the moment that the participants' concern for their understanding of the mathematical concepts involved begins to play a role in explaining choices made in discourse, then, I suggest, mathematical discourse may be said to exist. If we look at the various examples of discourse that may have originally been characterized as mathematical simply because they occurred in a mathematics classroom, included the use of mathematical vocabulary, or included mathematical content, we may find that the participants' concern for their own understanding of the mathematics is not always reflected in participants' discursive choices. In such a case, I would suggest that these types of discourse should not be considered mathematical.

Richards' (1991) four types of mathematical discourse may also be united by the factor of concern for one's own mathematical understanding, in terms of how the participants take care to either advance or correctly represent such understandings. In the discourse of research mathematicians, the advancement of mathematical understanding may be said to be the unifying goal of mathematical discourse. In the discourse of mathematical journals, it is not so much the development of understanding, but the communication of understanding in a clear and concise manner that determines organization, word choice, and other relevant decisions for discourse. While advancing mathematical understanding may not be a goal of the inquiry discourse of adults, efforts are made to correctly represent and apply mathematical understandings in ways that will solve problems. Ideally, the discourse of the mathematics classroom would be centered on the goal of advancing learners' mathematical understandings through instruction, and correctly representing that mathematical understanding for the purposes of assessment.

I suggest that mathematics educators define and study *mathematical* discourse as discourse in which the three explanatory factors of (1) experience and imagination, (2) social roles and responsibilities, and (3) concern for one's own understanding of the mathematics are reflected in participant choices. I believe that such a definition may assist both teachers and researchers in identifying why simply "having students talk to one another" while doing mathematics may not be a sufficient characterization of mathematics learning. For example, one may observe a mathematics classroom where the participants may (1) draw upon their experiences and imaginations, and (2) attend to their social roles and responsibilities as co-operative participants in discourse, but (3) lack in their efforts to either advance or express their own mathematical understandings. Classroom mathematical discourse must, by definition, be guided by the learners' concern for their own mathematical understanding. On the other hand, mathematical discourse may fall short for the purposes of learning mathematics if the participants (1) fail to connect their mathematical understanding to the relevant areas of personal experience or (2) fail to co-operatively attend to their social roles and responsibilities.

This suggested definition of mathematical discourse sets high standards for those who view mathematical discourse as central to mathematics learning. It implies that instructors should not only encourage students to draw upon their own experiences and engage in the social practices of questioning, explaining, and justifying, but should also seek to encourage these students to act based upon a concern for their own mathematical understanding. Determining how students may reach this point was not the focus of this study, although I believe that reviewing the data suggests that particular classroom norms enacted in the classroom in this study (for example, extended time to allow the negotiation of meaning to occur and the instructors' modeling of sincere questions about the intended meaning of the learners' language and gestures) and approaches to learning mathematics (such as the inclusion of contexts that allow participants multiple avenues to access discourse) certainly do not inhibit, and may be said to encourage, the exercise of personal agency in this way. Defining mathematical discourse has already entered a hermeneutic cycle of defining, applying, and redefining, and will likely continue in such a cycle as long as researchers are concerned with discourse in mathematics education. At present, though, I choose to define mathematical discourse as discourse in which the factors of personal experience, social awareness, and each individual's concern for their own understanding of the mathematics can be viewed as explanatory of human choice. This view of mathematical discourse invites researchers and practitioners to make choices that may increase the productive exercise of personal agency by learners of mathematics.

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APPENDIX A: THE QUABBIN RESERVOIR TASK

Quabbin Reservoir Task*

Name____

Please use additional paper as needed to provide complete answers.

The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph in Figure 6.38 represents the flow of water in and out of the Quabbin Reservoir throughout 1993.

(a) Sketch a possible graph for the quantity of water in the reservoir, as a function of time.

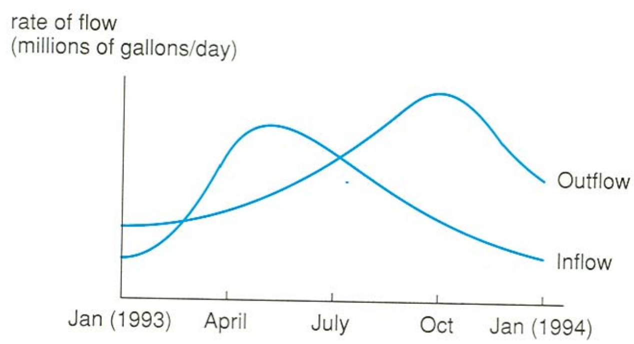

Figure 6.38

(b) Explain the changes in the quantity of water in the reservoir in terms of the relationships between outflow and inflow during each quarter of the year. How are these changes evident in your graph in part (a)?

(c) How does the quantity of water in the reservoir in Jan 1993 compare with the quantity of water in the reservoir in Jan 1994? How do you know?

*Adapted from Hughes-Hallett, et al. (1994). Calculus. New York, NY: John Wiley & Sons, Inc.

APPENDIX B: PARTICIPATION IN DISCOURSE

Participant	Number of clips for which the participant was the speaker
Daniel	284
Dr. Walter	33
Jamie	173
Julie	82
Justin	276

APPENDIX C: PRONOUN CODES

Pronouns	Number of Clips
Impersonal	
It	183
That	120
This	63
Those	3
These	4
Them-inanimate	5
They-inanimate	12
Personal	
He	2
I-personal	148
I-personifying	3
If you	22
She	2
They	7
We/us	110
You-one	87
You-plural	17
You-the hearer	70
You-personified	2
Possessive	80

APPENDIX D: VOCABULARY CODES

Code	Number of Clips
April	12
February	5
January	9
July	1
acceleration	2
add	7
after	5
again	5
all	3
amount	4
anti-derivative	13
apex	4
area	1
around	1
backwards	6
be	84
become	2
below	1
between	11
both	4
bottom	2
certain	2
change	28
chart	2
combine	4
come	21
concave	13
cone	3
consolidate	1
correspond	1
cross	1
cubic	3
cup	1
curve	5
cylinder	7
d-graph	13
dam	2
day	3
decrease	12
derivative	29
derive	1
difference	3
different	6
dimensional	2
direction	1
displacement	10
distance	6
down	22
drop	2
dy	1
end	1
enter	1
equal	13
equation	1
f of x	1
f prime of x	1
fall	2
fast	5
fill	1
first	18
five 5	1
flow	8
formula	1
fourth 4	3
frown	1
full	1
function	11
gallons	13
gallons per day	6
go	68
gradually	3
graph	48
great	10
gross	2
growth	1
half	1
height	5
high	11
hit	3
how much	5
idea	1
important	1

inflow	28
integral	3
irrelevant	1
keep	5
language	4
lateral	1
leave	6
left	1
less	7
level	15
line	7
little	5
low	5
magic	1
mark	1
maximum	5
measurement	6
meet	4
meter	2
middle	2
midpoint	6
minimum	1
minus	1
monkey wrench	2
more	7
most	3
negative	26
negativity	3
net	2
no	24
no change	3
normal	1
off	8
original	1
out	14
outflow	25
over time	4
overall	1
part	11
past	1
peak	1
period	2
pertinent	1
point	50
position	1
positive	16
positivity	2
problem	3
process	1
quantity	16
quarter	14
quick	1
rate	14
rate of change	22
rate of flow	15
reflect	1
relationship	1
remain	1
reservoir	10
result	1
right	61
rise	6
same	21
second 2	23
separate	6
shape	3
side	3
slide	1
slope	9
slow	3
small	2
speed	1
sphere	5
start	35
stay	3
still	11
stop	4
subtract	5
surface area	14
table	1
tangent	4
thickness	1
third 3	1
time	11
together	8
top	5
total	7
unit	4
up	18
velocity	37
volume	57
water	45

water level	19
weird	3
wording	1
x-axis	1
year	3
zero	29

APPENDIX E: CONCEPT CODES

Code	Description	Examples
Change in rate of change over time	Describes a change in the rate of change but does not quantify a rate.	“the negative slope starts going positive” “the slope starts becoming negative” “your speed would be increasing” “it [the water level] keeps rising faster and faster and faster” “increasing and increasing even less”
Change in volume over time	Describes a change in volume over a period of time, but does not attempt to quantify a rate.	“volume graph,” “changes in quantity of water,” “the quantity of water is increasing,”
Extrema	Extrema in either volume, rate of change, separate rates of change, or rate of rate of change	“highest velocity that you’ll have,” “lowest,” “most outflow,” “greatest slope is right here,” “highest point of flow rate,” “inflection point,” “top and bottom points,” “maximums and minimums,” “apex,” “peak”
Inflow = Outflow	Where the inflow is equal to the outflow	“where the outflow and the inflow meet,”
Net rate of change = 0	When the rate of change is zero/slope of the tangent line is zero	“the velocity hits zero,” “zero point,” “in a velocity graph it would be where it would cross the x-axis,” “the rate of change would be zero,” “where the tangent is zero”
no change in volume	When the volume is staying the same	“Leveling off” “no change in the water level for a certain time,” “the velocity of the water coming in equals the velocity of the water going out,”
Original graph	Verbal references or gestures that indicate all or parts of the given graph	“the inflow,” “Hector,” “these two functions as separate,” “we’re given this graph [pointing to the original graph] right here,” “those two graphs, adding them together,” “where they [inflow and outflow] meet,” “this [graph] is separate change”
Points and parts	References to specific points or periods, the horizontal axis or the vertical axis of a graph	Horizontal Axis: “time” “between,” “quarter,” “from here to here,” “April,” “parts,” “per day” Vertical Axis: “gallons,” “that point up there” and all references to extrema

Rate of change graph	Verbal references or gestures that indicate all or parts of the rate of change graph	"our velocity [graph]," "d-graph," "our rate graph," "what the derivative will look like" "the happy flow chart of our velocity"
Rate of change in volume	An instantaneous rate of change.	"how much the water level is changing," "the velocity of the flow of the water," "rate of flow"
Rate of change of rate of change	An attempt to quantify (as positive or negative) or label the rate of change of the rate of change.	"increasing much much faster positively" "the derivative of velocity is acceleration," "your rate of change of your rate of change" "concavity," "concave down," "concave up"
Rate of rate of change = 0	Points at which the concavity changes (down to up or vice versa) or extrema of the rate of change	"inflection point," "it starts concaving down," "the velocity is the highest"
Separate rates of change	Speaking of rate of change in terms of inflow or outflow	"velocity coming in," "velocity going out," "inflow," "outflow," "they're gradually becoming equal" "the same amount of water is coming in as it is leaving" "amount of water that is coming in"
Signs	Quantitative adjectives that come in opposite pairs that signify contrast within the task	"positive," "negative," "up," "down," "coming," "going"
Volume	The amount of volume in the reservoir, often at a specified point in time.	"the amount of water in the reservoir," "the quantity of water," "millions of gallons," "volume level," "the volume of the water," "displacement"
Volume graph	References to parts or all of the volume graph	"a displacement graph," "quantity of water going up or down," "volume graph," "this [graph] is kind of like our total inflow or outflow"

APPENDIX F: FULL TRANSCRIPT OF "THE GOSPEL ACCORDING TO JUSTIN"

330	(0:31:22.4)	Daniel:	Julie where you at?	
331	(0:31:23.5)	Justin:	How you doing Julie?	
332	(0:31:25.5)	Julie:	Um, I don't I don't know. I, I still don't understand where we're going.	
333	(0:31:32.9)	Justin:	What we're gonna do, let's see, is, this is the way I see it, alright? This is the gospel according to Justin.	
334	(0:31:40.7)	Jamie:	[laughs]	
335	(0:31:41.4)	Justin:	Kay, so we're given this, this graph right here right?	[Justin indicates the original graph on Julie's page. It is right side up for Julie, but upside down from his point of view]
			It gives us an outflow graph	[tracing outflow graph roughly from left to right with pencil tip]
			and an inflow graph.	[tracing inflow graph from right to left with pencil tip]
336	(0:31:48.0)	Julie:	Right.	
337	(0:31:48.5)	Justin:	Now, to me, you can't really do much when you want to know how, what the volume of the water is, with those two graphs separate.	
338	(0:31:54.2)	Justin:	So, what I'm thinking to do is to add them [*inflow and outflow*] together, so you take the difference between the two points, right?	
339	(0:32:03.3)	Justin:	So like, so you take the, you start, start with the income, uh, inflow I'm sorry, the inflow and you subtract the outflow from that part right, That's gonna give you the amount of water that's either	[tracing the vertical axis on the original graph between the inflow and the horizontal axis]
			"coming in" or "leaving," if it's negative it's leaving if it's positive it's, it's coming in.	[airquotes]

340	(0:32:23.7)	Justin:	Does that make sense?	
341	(0:32:24.3)	Julie:	Okay, yeah.	
342	(0:32:25.4)	Justin:	So, if you did that [*subtraction of outflow from inflow*] just over, you know, just did that for every single part, this would be that part that's leaving,	[shading the area between inflow and outflow from A to B]
			this is the, uh, water coming in, this is the, when the, um, water,	[the area between inflow and outflow from B to E]
			"volume" level is rising.	[airquotes]
			This is when it's [*the volume level*] going down again.	[the area between inflow and outflow from E to G]
343	(0:32:41.6)	Justin:	So basically, this	[Justin's rate graph in his notebook]
			is kind of what I did last time in class, it is, I kind of tried to sketch those two graphs adding them [*inflow and outflow*] together, so, go ahead.	
344	(0:32:49.8)	Julie:	So, like this minus this [*inflow minus outflow value between A and B*], wouldn't that make it [*the result of the subtraction, the value of the rate graph*] zero?	
345	(0:32:54.7)	Justin:	It [*the subtraction of inflow and outflow values indicated by Julie in 344*] would make it, well, it would make it negative, it would go below here, right? Cause if you take this distance right here, and you subtract this distance from that.	
346	(0:33:05.7)	Julie:	Oh, kay.	
347	(0:33:06.4)	Justin:	It's gonna put it [*the result of the subtraction in 344 as a value on the rate graph*]. down here someplace. Does that make sense?	
348	(0:33:08.8)	Julie:	Yeah.	

349	(0:33:09.9)	Justin:	And so it's [*the result of the subtraction in 344*] gonna give you a negative flow rate. Or in other words, the water, the, the "volume" of the water is lowering, right?	[airquotes]
350	(0:33:17.3)	Julie:	Okay.	
351	(0:33:18.3)	Justin:	So, this is, this is what I came up with.	[Justin's net rate graph in his notebook]
352	(0:33:20.9)	Justin:	So it's	[the rate graph at A]
			lower right here. Now it's [*the rate graph*] gradually coming to zero,	[the first intersection of the inflow and outflow on the original graph]
			it's gradually, um, coming to, it's gradually, **I don't want to say coming to zero**, I wanna say, [2 sec]	[the first x-intercept on Justin's net rate graph]
			yeah they're [*inflow and outflow*] gradually becoming equal, so it's the water level, the volume level is staying the same. Right?	[the first intersection of the inflow and outflow on the original graph] [holding hand flat with palm down at eye level]
353	(0:33:37.7)	Justin:	So at that point,	[first x-intercept on rate graph]
			this point right here, and at this point right here,	[the two points where the inflow and outflow intersect on the original graph]
			the same amount of water is coming in as it is leaving, right?	[moving both hands across the table at the same rate]
			is leaving, right?	[moving both hands across the table at the same rate]
354	(0:33:45.9)	Julie:	Right.	
355	(0:33:46.5)	Justin:	And so the water, volume of the water is gonna stay the same.	[holding arms out wide with palms in as if running them along the surface of a giant sphere]
			And so, it's um, the rate of change will be zero, does that make sense?	
356	(0:33:55.8)	Julie:	Cause if you subtracted this from this-	[pointing to the original graph at B]
357	(0:33:57.9)	Justin:	from this-	

358	(0:33:58.1)	Julie:	-it would be zero.	[touching horizontal axis of original graph at B]
359	(0:33:58.3)	Justin:	-it [*the result of the subtraction, the value of the rate graph*] would be zero, yeah.	[touching horizontal axis of original graph at B]
360	(0:33:59.8)	Justin:	And so, and then I just, I just kind of guess-timated the same thing. What's the distance between these, these two lines right here, like that.	[pointing to corresponding points on the inflow and outflow graphs between April and July]
361	(0:34:07.8)	Julie:	Uh-huh. So you're just taking this	[a point on the inflow graph between April and July]
			and subtracting this one,	[corresponding point on the outflow graph between April and July]
			which would put it like there-ish.	[a point in space on the original graph that is about the same distance above the x-axis as the difference between inflow and outflow for that same x value]
362	(0:34:12.2)	Justin:	Yeah, uh-huh.	
363	(0:34:13.8)	Julie:	Okay.	
364	(0:34:14.4)	Justin:	And then, um, doing the same thing from here,	[after the second intersection of inflow and outflow]
			but, sticking with the same one, taking the inflow	[inflow curve]
			and subtracting the outflow, even though the outflow is higher.	[outflow curve]
365	(0:34:21.7)	Julie:	Okay.	
366	(0:34:22.7)	Justin:	And so, that's kind of, how I'm looking at it, and so that kind of helps to combine the two graphs, like that, cause now you can see what the rate of flow, what the change of, in the flow rate, is, over time, and that kind of helps ya understand what's going on with the displacement graph.	[Justin traces his rate graph from right to left and then left to right three times as he speaks, finishing on the word "understand"]

			“Volume” graph.	[airquotes]
			Does that make sense?	
367	(0:34:40.1)	Julie:	Yeah.	
368	(0:34:40.8)	Justin:	Does that all make sense to you guys? Am I lying?	
369	(0:34:42.7)	Jamie:	So that’s, that’s your rate graph.	
370	(0:34:44.8)	Justin:	That’s that’s what I,	
			“imagine- “	[airquotes]
371	(0:34:46.7)	Jamie:	Okay.	[mimicking Justin’s airquotes]
372	(0:34:47.4)	Justin:	-picture it as.	
373	(0:34:49.0)	Jamie:	Yeah, it makes sense now.	
374	(0:34:50.3)	Justin:	And so, with this, is this kind of what your, your volume graph looks like?	[showing Daniel]
			Kind of, something kind of like that?	[Justin‘s volume graph]
375	(0:34:58.2)	Daniel:	Um, yeah.	
376	(0:35:01.1)	Justin:	Okay.	
377	(0:35:01.4)	Daniel:	You mean like this is the top of the dam?	[the horizontal axis on Justin’s volume graph]
378	(0:35:02.8)	Justin:	I don’t, that’s just uh, arbitrary [inaudible].	[traces the horizontal axis but then erases it]
379	(0:35:05.6)	Daniel:	Oh. Okay. Like it’s like that-ish.	[Daniel shows Justin his volume graph]
380	(0:35:09.8)	Justin:	It just all depends on where you start your water level at, yeah.	
381	(0:35:12.5)	Daniel:	Start at some *m*.	
382	(0:35:13.6)	Justin:	Yeah. So the same thing.	
383	(0:35:15.4)	Daniel:	Which stands for water level.	
384	(0:35:17.7)	Justin:	And so here	[Justin’s net rate graph before the first x-intercept]
			you know that your rate of change is negative, right?	
385	(0:35:21.3)	Daniel:	Yeah, so you’re going “doop, doop, doop, doop”	[falling intonation]
386	(0:35:23.6)	Justin:	And so you know your slope of your “volume graph,”	[airquotes] [Justin’s volume graph before the first minimum]
			the slope is going to be negative, right? Cause this part is negative.	[Justin’s net rate graph before the first x-intercept]
387	(0:35:30.2)	Justin:	And then you’re going to get	[first x-intercept on

			to this point of your, that it's [*the rate graph*] zero,	Justin's net rate graph]
			and so it's [*the volume graph*] going to level off right there, right?	[drawing a horizontal line at the first minimum on Justin's volume graph]
			There's going to be some point there where the tangent is zero.	[retracing the short horizontal line]
			Right? Does that make sense?	
388	(0:35:42.6)	Julie:	Yeah.	[really quiet, even for Julie]
389	(0:35:45.4)	Justin:	There's gonna be some point, 'cause, we're working backwards. Instead of finding the derivative,	[pointing to volume graph and sliding pencil up to the net rate graph]
			we're going from the derivative	[net rate graph]
			backwards.	[sliding pencil to volume graph]
			Trying to, trying to figure out how to go backwards, right?	[pointing first at the net rate graph and then sliding the point of the pencil to point at the volume graph again]
			And so if we have that point, zero on the derivative,	[first x-intercept on the net rate graph]
390	(0:35:54.9)	Justin:	that means that on the original graph,	[first minimum on the volume graph]
			that point is level, the tangent line is zero. There is no slope. Make sense?	
391	(0:36:07.3)	Julie:	Yeah.	
392	(0:36:08.3)	Justin:	So now, what's going to happen is the slope of this line is gonna start being positive.	[volume graph after the first minimum]
393	(0:36:13.2)	Justin:	Alright so it's [*the volume graph*] going to be concave up, like that,	[tracing the shape of the volume graph directly after the first minimum]
			until it [*the rate graph*] gets to its highest point of, um, flow rate.	[first maximum on the net rate graph]
394	(0:36:21.9)	Justin:	And then it's [*the rate*] gonna gradually still go, it's gonna gradually get, less positive.	[net rate graph as it decreases back to zero after the first maximum]

			It's [*the rate*] still positive see?	[lining up the length of the pencil with the horizontal axis and sliding it away from the axis in the positive vertical direction]
			The water level is still rising,	[pointing quickly to the volume graph after the first point of inflection and then quickly to the net rate graph after the first maximum]
			but it's [*the water level*] rising at a smaller rate. Does make sense?	
395	(0:36:31.5)	Jamie:	Yes.	
396	(0:36:32.5)	Daniel:	It's like our inflection point?	
397	(0:36:34.1)	Justin:	So. [1 sec] Let me finish explaining and then we'll go back.	[Julie looks at Daniel but then away]
398	(0:36:39.0)	Daniel:	Kay.	
399	(0:36:39.4)	Justin:	So it's [*the volume graph*] going concave up right here because it [*the rate graph*] keeps on getting higher and higher and higher	[after the first minimum on the volume graph] [after first x-intercept on net rate graph]
			and so it [*the volume graph*] keeps on raising faster and faster and faster	[approaching the first point of inflection on the volume graph]
			until it [*the volume graph*] gets to that inflection point, this point right here.	[first inflection point on volume graph, first maximum on rate graph]
400	(0:36:49.6)	Julie:	Okay.	
401	(0:36:50.6)	Justin:	And then all of the sudden, it's [*the volume graph*] still rising,	[right after the first maximum on the rate graph]
			but it's going, its rising gradually slower and so it [*the volume graph*] starts concaving down, right?	[right after the first inflection point on the volume graph]
402	(0:36:58.6)	Julie:	Okay.	
403	(0:36:59.0)	Justin:	And so, the water level is still rising,	[pointing pencil "upward" and pulling it quickly "upward"]
			it's just rising slower until it	[tracing concave down

			gets to that point right here,	area before maximum on volume graph]
			right here	[second x-intercept on rate graph]
			where it's [*the rate*] zero.	
404	(0:37:06.9)	Julie:	Kay so here	[first minimum on volume graph]
			and here	[first maximum on volume graph]
			correspond with-	
405	(0:37:10.6)	Justin:	With this point	[first x-intercept on rate graph]
			and that point	[second x-intercept on rate graph].
406		Julie:	-here	[pointing with Justin to the first x-intercept on rate graph]
			and here.	[pointing with Justin to the second x-intercept on rate graph]
407	(0:37:12.2)	Justin:	And this, um, point of inflection,	[first inflection point on the volume graph]
			what we're calling the point of inflection is this point right here.	[first maximum on rate graph]
			Does that make sense?	
408	(0:37:20.0)	Julie:	[2 sec] Yes.	
409	(0:37:22.9)	Justin:	Okay. So, then what happens? We come, start coming, we're going negative,	[tracing pencil point along rate curve after the second x-intercept]
			right? Increasing the negativity of that.	[Justin's rate graph from second intercept to the minimum, E-F]
410	(0:37:32.6)	Justin:	So it [*water level/the volume graph*] keeps going down, faster and faster	[tracing volume graph after the first maximum towards second minimum]
			and faster because the velocity is getting lower and lower and lower right?	[tracing rate graph from second x-intercept to the minimum]
411	(0:37:38.2)	Justin:	So it keeps going down	[tracing volume graph from first maximum toward second minimum]
			until it gets to some point right here	[minimum on rate graph]

			where it's still gonna be negative,	[tracing back and forth on rate graph after the minimum]
			but it's gonna start coming, be less negative, if that makes sense.	[lowering hand in the air]
412	(0:37:47.7)	Julie:	Yeah.	
413	(0:37:48.0)	Justin:	It's kind, it's [*411*] a really bad way to say it [*the negative rate is increasing on F to G*], but that's the only thing I can think of.	
414	(0:37:50.4)	Julie:	Okay.	
415	(0:37:50.4)	Justin:	And that's [*point F on the volume graph*] where your point of inflection is.	[pointing to the volume graph, although the exact point is not clear]
416	(0:37:52.3)	Julie:	Okay.	
417	(0:37:53.7)	Justin:	Right? Because it's still, it's going negative really fast, really fast, really fast, really fast until gets to this point where it [*the rate*] starts, where it stops	[rate graph and volume graph approaching the inflection point]
418	(0:38:01.7)	Julie:	[inaudible, "Increasing?"]	
419	(0:38:02.3)	Justin:	increasing, um-	[moving hand toward Julie]
420	(0:38:04.5)	Julie:	[inaudible, "negative, so it's decreasing?"]	
421	(0:38:05.4)	Justin:	-it stops decreasing	[pointing to the rate graph]
			and it starts, the velocity starts being,	[pulling hand toward self along the table top]
			yeah, and so it [*volume graph*] kind of starts leveling out. Does that make sense?	[volume graph after the inflection point]
422	(0:38:12.5)	Justin:	So if this line were to keep going up here to zero,	[rate graph after the minimum]
			it'd [*volume graph*] keep going around like this until-	[continuing the concave up curve on the volume graph after the inflection point]
423	(0:38:17.0)	Julie:	Oh, okay.	
424	(0:38:17.4)	Justin:	-you get that point.	[pointing the minimum that he has just drawn on the volume graph]

425	(0:38:18.5)	Justin:	Does that make sense?	[erasing the curve that he just extended]
426	(0:38:19.6)	Julie:	Yeah.	
427	(0:38:20.3)	Justin:	So that's kind of how I've been looking at it [*the Quabbin Reservoir Task*].	
428	(0:38:22.2)	Julie:	Kay.	
429	(0:38:25.2)	Justin:	Now the thing you don't know, which is like kind of what Daniel said,	[pointing toward Daniel with his pencil]
			cause I had that line right here.	[the horizontal line that Justin previously erased from his volume graph]
			You don't know where to start this graph at.	[touching points on the vertical axis of the volume graph]
430	(0:38:31.8)	Justin:	I mean, this graph, you know, this could be one thousand, um, cubic gallons, or whatever I don't know you say that.	[the y-intercept of the volume graph]
431	(0:38:38.2)	Justin:	Or it [*the initial value of the volume graph*] could be, you know, you don't know how, where to start this graph at.	[holding pencil parallel to the horizontal axis of the volume graph and sliding it back and forth in the "vertical" direction]
432		Jamie:	Okay.	
433	(0:38:42.3)	Justin:	Does that make sense?	
434	(0:38:43.5)	Julie:	Mmhmm.	
435	(0:38:46.1)	Justin:	But you can kind of get the general idea, that, from working backwards.	[pointing to the rate graph and sliding the point of the pencil down the page to point at the volume graph]
			Of what it's [*the volume graph*] gonna look like.	
436	(0:38:51.3)	Julie:	Okay.	
437	(0:38:52.9)	Justin:	But does that actually, before we get to that, does that make sense?	
			how I went from, how I combined the two graphs	[pointing to the rate graph]
			and then how I went, how I looked at this graph	[rate graph]
			and then tried to make a,	[volume graph]

			um volume. Take away my "water level,"	[erasing a mark on the volume graph]
			that's right this [*the label on the vertical axis of the volume graph*] should be "volume."	
438	(0:39:08.1)	Julie:	[5 sec] Is it volume?	
439	(0:39:13.4)	Justin:	Yeah, 'cause we're working, it'll [*the volume graph*] be the volume of the water.	[pointing to the volume graph]
440	(0:39:16.8)	Justin:	'Cause this, the rate is in gallons, gallons per day.	[the rate graph]
441	(0:39:21.1)	Justin:	[3 sec] And so you just want to keep the same kind of units so it'd be gallons, cubic gallons, I guess is what it would be.	[pointing to the volume graph]
442	(0:39:28.9)	Julie:	Oh. Oh, okay, yeah.	
443	(0:39:31.5)	Justin:	So volume.	
444	(0:39:32.7)	Justin:	So does that make sense?	[circling the page with the rate and volume graphs with the pencil tip]
445	(0:39:36.3)	Julie:	Yeah.	
446	(0:39:37.4)	Justin:	Do you have any questions?	
447	(0:39:38.4)	Julie:	No, I think that's, just like, the top and the bottom points are your zero points?	[pointing to points on her page]
448	(0:39:45.7)	Justin:	On, top and your bottom points for what graph?	
449	(0:39:49.6)	Julie:	Like these points on your volume graph	[minimum and maximum on her volume graph]
			are your zero points on your velocity?	[touching her rate graph]
450	(0:39:56.2)	Justin:	Yes.	[nodding]
451	(0:39:56.9)	Daniel:	volume of a sphere [inaudible]	
452	(0:40:01.2)	Justin:	And then your maximums and your minimums on your veloc-, your, we'll call	[Julie's rate graph]
			"velocity"	[airquotes]
			graph are going to be your points of inflection on your displacement graph.	[pointing to Julie's volume graph]